Student Name: ____________________

Student Number: ____________________

Total Marks: ________
100

Okanagan University College
Final Examination

Math 112 (Fall, 2001)
Solutions

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Section(s): **01, 02, 51, 61, 71, & 72**

001214

1:00PM

Duration: **3 hours**

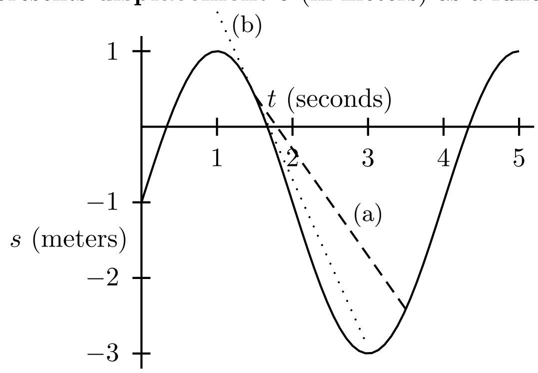
READ INSTRUCTIONS CAREFULLY BEFORE COMMENCING EXAM

INSTRUCTIONS: Answer all 13 questions in the spaces provided, showing all significant steps. Partial marks will be awarded for correct work even if the final answer is incorrect. Marks per question are given in the left margin, total 100. Check that your paper contains all 9 pages in addition to the cover page.

This paper contains pages numbered 1 to 9

EXAM BOOKLETS ARE NOT REQUIRED

1 The following graph represents **displacement** s (in meters) as a function of **time** t (in seconds).



[2] (a) Use the graph to **estimate** the **average velocity** in the time interval $1.5 \le t \le 3.5$. Show your calculations and give your answer to one figure to the right of the decimal place. The average velocity is represented by the slope of a certain line. Draw this line (as a dashed line) on the graph.

$$\overline{v} \approx \frac{-2.2 - 0.6}{3.5 - 1.5} = -\frac{2.8}{2} = -1.4\,\text{m/s}$$

The secant line is the dashed line on the graph above, labelled (a).

[2] (b) Use the graph to **estimate** the **instantaneous velocity** at time $t = 1.5$. Show your calculations and give your answer to one figure to the right of the decimal place. The instantaneous velocity is also represented by the slope of a certain line. Draw this line (as a solid line) on the graph.

$$v \approx \frac{-2.5 - 1.5}{3 - 1} = -\frac{4}{2} = -2\,\text{m/s}$$

The tangent line is the dotted line on the graph above, labelled (b).

[2] (c) Are there any times from $t = 0$ to $t = 4$ when the **instantaneous velocity** is zero? If so, what times?

The times when the instantaneous velocity is zero are the times when the tangent line is horizontal. This takes place when $t = 1$ and $t = 3$.

[1] (d) There is an inflection point at $t = 2$. What can you say about the instantaneous velocity at $t = 2$?

At the inflection point at $t = 2$ the instantaneous velocity has its most negative value, or in other words, it changes from decreasing to increasing.

2 Calculate the following **derivatives**, **simplify only if instructed to do so**:

[3] (a) $h'(3)$, given $h(t) = t^2 \ln(t+2)$.

$$h'(t) = 2t\ln(t+2) + t^2\left(\frac{1}{t+2}\right) = 2t\ln(t+2) + \frac{t^2}{t+2} \Rightarrow h'(3) = 6\ln 5 + \frac{9}{5} = 11.46$$

[3] (b) $f'(x)$, given $f(x) = 1 + x + \arctan\left(\frac{x}{2}\right)$. **Simplify** your answer.

$$f'(x) = 1 + \frac{1}{1+\left(\frac{x}{2}\right)^2}\left(\frac{1}{2}\right) = 1 + \frac{2}{4+x^2} = \frac{6+x^2}{4+x^2}$$

[3] (c) $\frac{dy}{dx}$, given $y = \left[\sqrt{x^3+2} + \cos(2x)\right]^6$.

$$\begin{aligned}\frac{dy}{dx} &= 6\left[\sqrt{x^3+3} + \cos(2x)\right]^5 \left[\left(\frac{1}{2}\right)\left(x^3+2\right)^{-1/2}\left(3x^2\right) - 2\sin(2x)\right] \\ &= 6\left[\sqrt{x^3+3} + \cos(2x)\right]^5 \left(\frac{3x^2}{\sqrt{x^3+2}} - 2\sin(2x)\right)\end{aligned}$$

[3] (d) $\frac{dI}{dL}$, given $I = \frac{V}{\sqrt{R^2+\omega^2 L^2}}$ where V, R and ω are constants.

$$\begin{aligned}\frac{dI}{dL} &= V\left[\left(-\frac{1}{2}\right)\left(R^2+\omega^2L^2\right)^{-3/2}\left(\omega^2(2L)\right)\right] \\ &= -\frac{\omega^2 VL}{\left(R^2+\omega^2L^2\right)^{3/2}}\end{aligned}$$

[4] (e) $k''(z)$, given $k(z) = \frac{3z-2}{5z+4}$. **Simplify** your answer.

$$\begin{aligned}k'(x) &= \frac{3(5z+4) - 5(3z-2)}{(5z+4)^2} = \frac{15z+12-15z+10}{(5z+4)^2} = \frac{22}{(5z+4)^2} \\ k''(x) &= \frac{22(-2)(5)}{(5z+4)^3} = -\frac{220}{(5z+4)^3}\end{aligned}$$

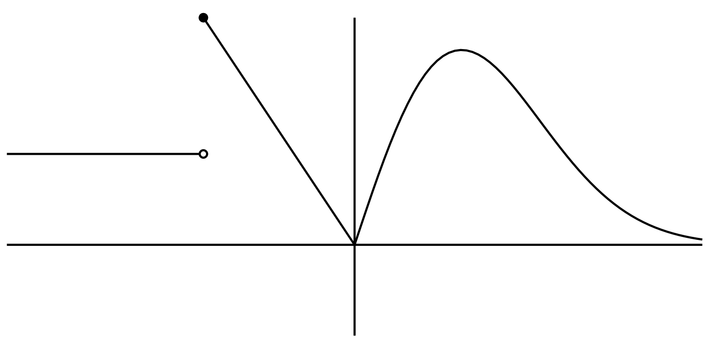
[4] (f) $\frac{dy}{dx}$, given $e^{(x^2)}y^3 = 4 + \sin(y)$.

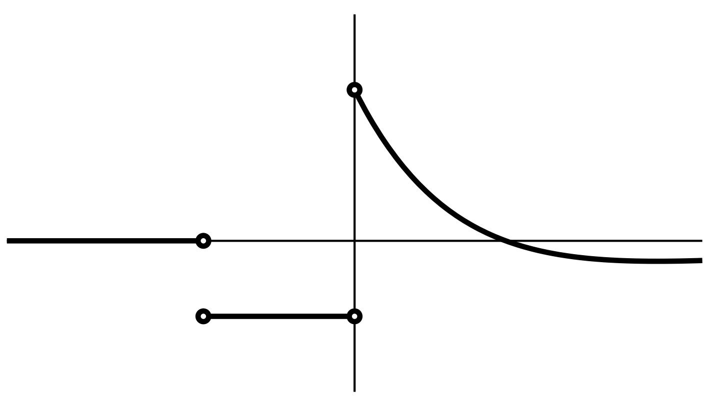
$$\begin{aligned}2xe^{(x^2)}y^3 + e^{(x^2)}\left(3y^2y'\right) &= \cos(y)\,y' \\ \left[\cos(y) - 3y^2e^{(x^2)}\right]y' &= 2xe^{(x^2)}y^3\end{aligned}$$

[5] 3 Given $f(x) = \dfrac{x}{x-2}$, use the **limit definition** of the derivative to show that $f'(x) = \dfrac{-2}{(x-2)^2}$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-2} - \dfrac{x}{x-2}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h}{x+h-2} - \frac{x}{x-2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-2) - x(x+h-2)}{(x+h-2)(x-2)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2-2x+xh-2h-\left(x^2+xh-2x\right)}{(x+h-2)(x-2)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2-2x+xh-2h-x^2-xh+2x}{(x+h-2)(x-2)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-2h}{(x+h-2)(x-2)}\right) = -\lim_{h\to 0} \frac{2}{(x+h-2)(x-2)} \\
&= -\frac{2}{(x-2)^2}
\end{aligned}
$$

[4] 4 The following is the graph of some function $f(x)$. **Sketch the graph of the derivative function** $f'(x)$ as accurately as you can on the set of coordinate axes provided.





5 Let $g(x) = x\,2^{h(x)}$, where $h(3) = -2$ and $h'(3) = 5$.

[4] (a) **Find an expression for** $g'(x)$ in terms of $h(x)$ and $h'(x)$.

$$g'(x) = 2^{h(x)} + x2^{h(x)} \ln 2h'(x) = 2^{h(x)}\left(1 + xh'(x)\ln 2\right)$$

[2] (b) **Give the exact value** of $g'(3)$, not a decimal approximation. Your value will contain $\ln 2$.

$$g'(3) = 2^{h(3)}\left(1 + 3h'(3)\ln 2\right) = 2^{-2}\left(1 + 3 \cdot 5 \cdot \ln 2\right) = \frac{1}{4}\left(1 + 15\ln 2\right)$$

[5] 6 Consider the limit

$$\lim_{x\to 3} \frac{x^2 + x + 2a}{x^2 - x - 6}$$

Find a value of a for which this **limit exists** and find the **value of the limit** for this a.

Note that

$$x^2 - x - 6 = (x-3)(x+2)$$

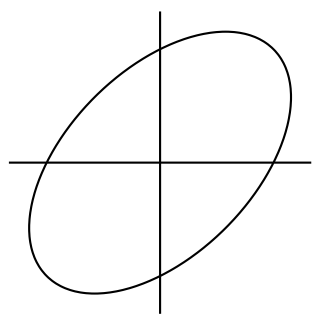
So that we must find a so that the numerator of the fraction in the limit is zero at $x = 3$. This gives

$$3^2 + 3 + 2a = 0 \Rightarrow a = -6$$

Then

$$\lim_{x\to 3} \frac{x^2 + x - 12}{x^2 - x - 6} = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+4}{x+2} = \frac{7}{5}$$

7 The graph of the equation $x^2 - xy + y^2 = 9$ is the "tilted" ellipse shown in the following diagram.



[1] (a) What are the **coordinates of the two points** where the ellipse intersects the x-axis?

The x-intercepts are where $y = 0$. This gives

$$x^2 - x(0) + 0^2 = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

The coordinates of the two x-intercepts of the ellipse are $(3, 0)$ and $(-3, 0)$.

[3] (b) Use implicit differentiation to **find** $\dfrac{dy}{dx}$.

$$2x - y - xy' + 2yy' = 0 \Rightarrow (2y - x)\,y' = y - 2x \Rightarrow y' = \frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

[2] (c) Determine the equations of the **two tangent lines** to the ellipse at the points in part (a).

At $(3,0)$ the slope is

$$m = \left.\frac{dy}{dx}\right|_x = 3, y = 0 = \frac{0-2(3)}{0-3} = 2$$

The equation of the tangent line is

$$y - 0 = 2(x-3) = 2x - 6$$

At $(-3,0)$ the slope is

$$m = \left.\frac{dy}{dx}\right|_x = 3, y = 0 = \frac{0-2(-3)}{0-(-3)} = 2$$

The equation of the tangent line is

$$y - 0 = 2(x+3) = 2x + 6$$

[1] (d) Do these two tangent lines intersect? If so, find the coordinates of the point of intersection. If not, briefly explain why not.

The tangent lines do not intersect. They are parallel since they have the same slope.

[4] 8 **Verify** that the equation $x + \sin x = 3$ has at least one real solution. Show all work and justify your conclusion.

Let $f(x) = x + \sin x - 3$. Then any solution to the equation above is an x value for which $f(x) = 0$. This function is continuous for all x, so we can apply the Intermediate Value Theorem. Further, we have

$$f(0) = 0 + \sin 0 - 3 = -3 < 0 \quad \text{and} \quad f(\pi) = \pi + \sin \pi - 3 = \pi - 3 > 0$$

So by the Intermediate Value Theorem there is a number c in the interval $(0, \pi)$ for which $f(c)$. This c is a solution to the original equation.

9 Consider the function $f(x) = \sqrt[5]{2-x}$.

[3] (a) Find the **tangent line approximation** (the **linearization**) of $f(x)$ at $x = 1$.

$$f'(x) = \frac{1}{5}(2-x)^{-4/5}(-1) = -\frac{1}{5(2-x)^{4/5}}$$

Thus, $f(1) = -\frac{1}{5}$. So the linearization at $x = 1$ is

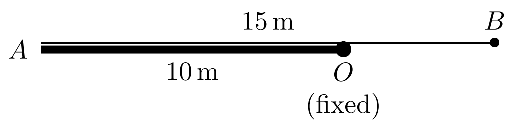
$$f(x) \approx L(x) = f(1) + f'(1)(x-1) = 1 - \tfrac{1}{5}(x-1)$$

[2] (b) Use your result in part (a) to **estimate** $\sqrt[5]{1.01}$.

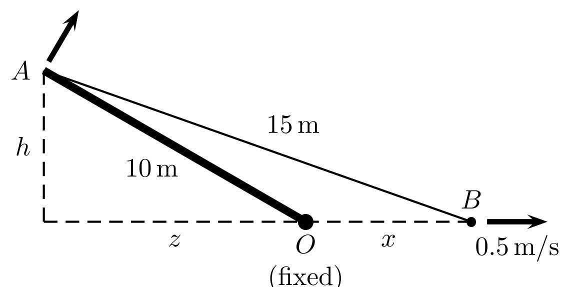
To use the function f compute $\sqrt[5]{1.01}$ we must use $x = 0.99$, that is, $f(0.99) = \sqrt[5]{2-0.99} = \sqrt[5]{1.01}$. By the tangent line approximation above

$$\sqrt[5]{1.01} = f(0.99) \approx L(0.99) = 1 - \tfrac{1}{5}(0.99-1) = 1 - \tfrac{1}{5}(-0.01) = 1.002$$

10 A 10-metre totem pole will be raised by fixing one end to the ground at point O in the diagram, and attaching a 15-metre rope to the other end, A. The rope is laid out along the totem pole and extended beyond the fixed end at O to the point B.



The end at B is then pulled along the ground at 0.5 m/s directly away from the fixed end of the totem pole raising the end A off the ground.



[3] (a) Use Pythagoras theorem for each of the two right triangles in the second diagram to **express z in terms of x.**

$$h^2 + z^2 = 100 \Rightarrow h^2 = 100 - z^2$$

and

$$(z+x)^2 + h^2 = 15^2 = 225 \Rightarrow z^2 + 2xz + x^2 + 100 - z^2 = 225 \Rightarrow 2xz + x^2 = 125$$

[2] (b) Give **the values of x and z** when the totem **pole is on the ground** and **ten seconds after the rope starts moving**.

When $t = 0$, we have $x = 5$ and $z = 10$, since at this time $h = 0$. When $t = 10$ we have $x = 5 + 0.5(10) = 10$ and

$$2z(10) + 100 = 125 \Rightarrow 20z = 25 \Rightarrow z = \tfrac{5}{4} = 1.25$$

[4] (c) Assuming that the sun is directly overhead, find the **rate at which the length of the shadow** cast by the totem pole is changing ten seconds after the rope starts moving.

If the sun is directly overhead, then z is the length of the shadow. Thus, we are given $\frac{dx}{dt} = 0.5$ and we are asked to find $\frac{dz}{dt}$ when $x = 10$ and $z = 1.25$. Taking derivative of the second relation found in part (a) with respect to t gives

$$2x\frac{dz}{dt} + 2z\frac{dx}{dt} + 2x\frac{dx}{dt} = 0 \Rightarrow \frac{dz}{dt} = -\left(1 + \frac{z}{x}\right)\frac{dx}{dt}$$

When $x = 10$, $z = 1.25$, and $\frac{dx}{dt} = 0.5$ we have

$$\frac{dx}{dt} = -\left(1 + \frac{1.25}{10}\right)(0.5) = -1.125(0.5) = -0.5625\,\text{m/s}$$

11 **Evaluate each limit**. You may use l'Hopital's rule where appropriate.

[3] (a) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{1 - \cos(2x)}$

$$\begin{aligned} \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(2x)} &= \frac{0}{0} \\ &\overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin(2x)} = \frac{0}{0} \\ &\overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{4\cos(2x)} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$

[3] (b) $\lim\limits_{x\to\infty} \dfrac{1 - \ln x}{1 + e^{-x}}$

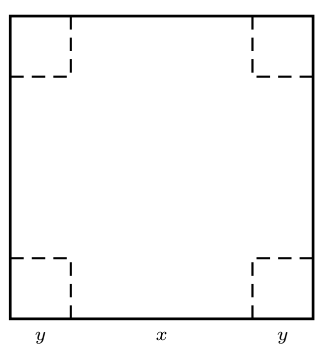
Note that $\ln x \to \infty$ as $x \to \infty$ so that $1 - \ln x \to -\infty$ as $x \to \infty$. Further, $e^{-x} \to 0$ as $x \to \infty$ so that $1 + e^{-x} \to 1$ as $x \to \infty$. Thus

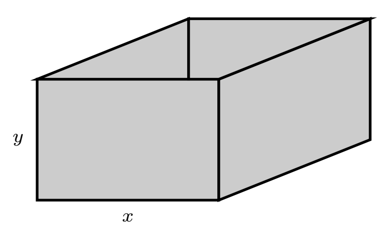
$$\lim_{x\to\infty} \frac{1 - \ln x}{1 + e^{-x}} = -\infty$$

[3] (c) $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$

$$\begin{aligned} \lim_{x\to 0^+} \sqrt{x}\ln x &= 0 \cdot \infty \\ &= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \frac{\infty}{\infty} \\ &\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{(-1/2)\,x^{-3/2}} = \lim_{x\to 0^+} -2x^{1/2} = 0 \end{aligned}$$

[7] 12 A square sheet of paper is to be made into a container by cutting squares out of each of the corners. The container is to have a square base, vertical sides and an open top with a volume of exactly 2000cm^3. **Determine the dimensions of the container** so that the total area of the paper, including the discarded square corners, is minimum. See the diagram. Hint: Can you just minimize the length of one side of the original square piece of paper?





Let x be the length of the sides of the square base of the box, and let y be its height. See the diagrams above. Then the surface area of the box is

$$A = (x + 2y)^2$$

We wish to find the values of x and y so that this function is minimum. Since we have a function of two variables, we must eliminate one of the variables using a constraint. Here the constraint comes from the volume of the box which is

$$V = x^2 y = 2000 \Rightarrow y = \frac{2000}{x^2}$$

So that

$$A = \left(x + \frac{4000}{x^2}\right) \Rightarrow \sqrt{A} = x + \frac{4000}{x^2}$$

We now wish to find x, for $x > 0$, so that A, or equivalently $\sqrt{A}$, is minimum.

First find any critical numbers. The derivative is

$$\frac{d}{dx}\sqrt{A} = 1 - \frac{8000}{x^3} = \frac{x^3 - 8000}{x^3}$$

so that $\frac{d}{dx}\sqrt{A} = 0$ gives

$$x^3 - 8000 = 0 \Rightarrow x = 20$$

Next check that this critical number gives a minimum. Use the second derivative test:

$$\frac{d^2}{dx^2}\sqrt{A} = \frac{2400}{x^4} > 0 \quad \text{for any } x > 0$$

Since the second derivative is positive, the critical number gives a local minimum. Further, since this is the only critical number in the domain of the function, it must give the absolute minimum.

Finally, find y. For $x = 20$

$$y = \frac{2000}{20^2} = \frac{2000}{400} = 5$$

Thus, for minimum surface area with a volume of $2000\,\text{cm}^3$ the box should have a base 20 cm on a side and a height of 5 cm.

13 A certain function f has domain $(-\infty, -1) \cup (-1, \infty)$ and is continuous on its domain. Further,

$$f(-3) = 8,\ f(2) = 3,\ f(3) = 4$$

[3] (a) The function f described above satisfies:

$$f'(x) > 0 \text{ for } x < -3 \text{ and } x > 2$$
$$f'(x) < 0 \text{ for } -3 < x < -1 \text{ and } -1 < x < 2$$
$$f'(-3) \text{ does not exist}$$

Give the **intervals where f is increasing and decreasing**. Identify all of the **critical numbers** and classify each as a **local maximum, local minimum, or neither**.

From the given information we see that

$$\begin{array}{lll} f \text{ is} & \text{increasing} & \text{on } (-\infty, -3) \cup (2, \infty) \\ f \text{ is} & \text{decreassing} & \text{on } (-3, -1) \cup (-1, 2) \end{array}$$

From this we see that at the critical number at $x = -3$ there is a local maximum, since f changes from increasing to decreasing there; and at the critical number at $x = 2$ there is a local minimum, since f changes from decreasing to increasing there.

[3] (b) The same function f also satisfies:

$$f''(x) > 0 \text{ for } x < -3 \text{ and } -1 < x < 3$$
$$f''(x) < 0 \text{ for } -3 < x < -1 \text{ and } x > 3$$
$$f''(-3) \text{ does not exist}$$

Give the intervals where the graph of f is **concave up and concave down**. Identify all **inflection points**. From the given information we see that

$$\begin{array}{lll} \text{graph of } f \text{ is} & \text{concave up} & \text{on } (-\infty, -3) \cup (-1, 3) \\ \text{graph of } f \text{ is} & \text{concave down} & \text{on } (-3, -1) \cup (3, \infty) \end{array}$$

From this we see that there are inflection points at $x = -3$ and $x = 3$. Note that there is not an inflection point at $x = -1$, since the function f is not defined at $x = -1$.

[3] (c) The same function f satisfies:

$$\lim_{x\to\infty} f(x) = 5 \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$
$$\lim_{x\to-1^-} f(x) = -\infty \text{ and } \lim_{x\to-1^+} f(x) = \infty$$

Using this information together with any relevant information from parts (a) and (b) above determine any **vertical or horizontal asymptotes** of the graph of f. **Describe the behaviour** of the function f near any **vertical asymptotes** and the **end behaviour of the function** for both large positive and large negative x.

Since $\lim_{x\to\infty} f(x) = 5$ there is a horizontal asymptote at $y = 5$ on the right. Since $\lim_{x\to-\infty} f(x) = -\infty$ there is no horizontal asymptote on the left.

Since $\lim_{x\to-1^-} f(x) = -\infty$ and $\lim_{x\to-1^+} f(x) = \infty$, there is a vertical asymptote at $x = -1$. From the left the graph goes down the vertical asymptote and from the right the graph goes up the vertical asymptote.

[3] (d) **Sketch a possible graph of** f whose properties are given above. Label each **critical number** and **inflection point** with its correct coordinates. Identify any **vertical and/or horizontal asymptotes**.