Student Name: ____________________

Student Number: ____________________

Total Marks: ________
100

Okanagan University College
Final Examination – Solutions

Math 112 (Fall, 2002)

Instructor(s): Clint Lee

Section(s): 71 & 72

December 17, 2002

9:00AM

Duration: 3 hours

READ INSTRUCTIONS CAREFULLY BEFORE COMMENCING EXAM

INSTRUCTIONS: Answer all 15 questions in the spaces provided, showing all significant steps. Partial marks will be awarded for correct work even if the final answer is incorrect. Marks per question are given in the left margin, total 100. Check that your paper contains all 7 pages in addition to the cover page.

This paper contains pages numbered 1 to 7

EXAM BOOKLETS ARE NOT REQUIRED

1 For each of the following, either evaluate the limit, giving the exact value or $+\infty$ or $-\infty$, or show that the limit does not exist. Show all your calculations and justify all your claims.

[2] (a) First factor the denominator and cancel $x-2$. Then substitute $x=2$.

$$\lim_{x\to 2}\frac{x-2}{x^3+3x^2-10x} = \lim_{x\to 2}\frac{x-2}{x(x+5)(x-2)} = \lim_{x\to 2}\frac{1}{x(x+5)} = \frac{1}{14}$$

[2] (b) Substitute $x=1$.

$$\lim_{x\to 1}\frac{\arcsin x+\arctan x}{x+1} = \frac{\frac{\pi}{2}+\frac{\pi}{4}}{2} = \frac{3\pi}{8}$$

[2] (c) Substituting $x=1$ gives $\frac{1}{0}$, thus this is an infinite limit. For $x<1$, $\ln x<0$ and near $x=1$, of course, $x>0$. So

$$\lim_{x\to 1^-}\frac{x}{\ln(x)} = -\infty$$

[2] (d) Since $-1\le \sin\left(x^2\right)\le 1$ we have $\left|\frac{\sin(x^2)}{x}\right| \le \frac{1}{x}\to 0$ as $x\to\infty$. Thus,

$$\lim_{x\to\infty}\frac{\sin(x^2)}{x} = 0$$

[4] 2 (a) Given $f(x)=\frac{4}{x+2}$, use the **limit definition** of derivative to show that $f'(x)=\frac{-4}{(x+2)^2}$.

By the definition of the derivative

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(f(x+h)-f(x)\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{4}{x+h+2}-\frac{4}{x+2}\right) = \lim_{h\to 0}\frac{4}{h}\left(\frac{x+2-x-h-2}{(x+2)(x+h+2)}\right) \\ &= \lim_{h\to 0}\frac{4}{h}\left(\frac{-h}{(x+2)(x+h+2)}\right) = \lim_{h\to 0}\frac{-4}{(x+2)(x+h+2)} = -\frac{4}{(x+2)^2} \end{aligned}$$

[2] (b) Use the result of part (a) to find the equation of the tangent line to the graph of $f(x)=\frac{4}{x+2}$ at the point where $x=1$.

When $x=1$ the y-coordinate of the point on the graph of the f is $y=f(1)=\frac{4}{3}$ and the slope of the tangent line is $m=f'(1)=-\frac{4}{3^2}=-\frac{4}{9}$. So the equation of the tangent line at the point $\left(1,\frac{4}{3}\right)$ is

$$y-2=-\frac{4}{9}\left(x-\frac{4}{3}\right) \Rightarrow y=-\frac{4}{9}x+\frac{7}{27}$$

3 Suppose that the function f is given by the following table of values.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	2.31	2.36	2.40	2.42	2.50	2.54	2.49

[2] (a) Compute the **average** rate of change of f between $x=0.0$ and $x=3.0$.

The average rate of change is

$$\text{average rate of change} = \frac{f(3.0)-f(0.0)}{3.0-0.0} = \frac{2.49-2.31}{3.0} = \frac{0.18}{3.0} = 0.06$$

[2] (b) **Estimate** $f'(1.5)$. Give the best numerical estimate you can.

The best estimate of the value of the derivative at $x=1.5$ is to use the values on either side of $x=1.5$. Thus,

$$f'(1.5) \approx \frac{f(2.0)-f(1.0)}{2.0-1.0} = \frac{2.50-2.40}{1.0} = 0.10$$

4 Find the indicated derivative(s) of each function. You may use logarithmic differentiation if you think it will be helpful.

[3] (a) $f(x) = (\sin(\cos 4x))^3$, find $f'(x)$

Applying the chain rule four times gives

$$f'(x) = 3(\sin(\cos 4x))^2(\cos(\cos 4x))(-\sin 4x)(4) = -12\sin 4x\cos(\cos 4x)(\sin(\cos 4x))^2$$

[4] (b) $y = x\arctan(2x)$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

For $\dfrac{dy}{dx}$ use the product rule followed by the chain rule in the second term:

$$\frac{dy}{dx} = \arctan(2x) + x\left(\frac{1}{1+(2x)^2}\right)(2) = \arctan(2x) + \frac{2x}{1+4x^2}$$

For $\dfrac{d^2y}{dx^2}$ use the chain rule on the first term above and the quotient rule on the second term

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{2}{1+4x^2} + \frac{2\left(1+4x^2\right) - 2x(8x)}{\left(1+4x^2\right)^2} \\ &= 2\left(\frac{1+4x^2+1+4x^2-8x^2}{\left(1+4x^2\right)^2}\right) = \frac{4}{\left(1+4x^2\right)^2}\end{aligned}$$

[4] (c) $p(r) = \sqrt{\dfrac{r^2+1}{r^2-1}}$, find $p'(r)$

Applying the chain rule followed by the quotient rule gives

$$\begin{aligned}p'(r) &= \frac{1}{2}\left(\frac{r^2+1}{r^2-1}\right)^{-1/2}\left(\frac{2r\left(r^2-1\right) - \left(r^2+1\right)(2r)}{\left(r^2-1\right)^2}\right) \\ &= \frac{\left(r^2-1\right)^{1/2}}{\left(r^2+1\right)^{1/2}}\left(\frac{r^3-r-r^3-r}{\left(r^2-1\right)^2}\right) \\ &= \frac{-2r}{\left(r^2-1\right)^{3/2}\left(r^2+1\right)^{1/2}}\end{aligned}$$

[3] (d) Given: $G(t) = f(h(t))$, $h(1) = 4$, $f'(4) = 3$, and $h'(1) = -6$. Find $G'(1)$.

By the chain rule

$$G'(t) = f'(h(t))h'(t)$$

so that

$$G'(1) = f'(h(1))h'(1) = f'(4)h'(1) = (3)(-6) = -18$$

[4] 5 Use logarithmic differentiation to find $f'(x)$ for $f(x) = x^{(\sin x + \cos x)}$.

Let $y = x^{(\sin x+\cos x)}$, then taking the natural logarithm gives

$$\ln y = (\sin x + \cos x)\ln x$$

Then, applying the product rule gives

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= (\cos x - \sin x)\ln x + (\sin x + \cos x)\left(\frac{1}{x}\right) \\ &= \cos x\ln x - \sin x\ln x + \frac{\sin x}{x} + \frac{\cos x}{x} \\ &= \cos x\left(\frac{1}{x} + \ln x\right) + \sin x\left(\frac{1}{x} - \ln x\right)\end{aligned}$$

Solving for the derivative gives

$$\frac{dy}{dx} = x^{(\sin x+\cos x)}\left[\cos x\left(\frac{1}{x} + \ln x\right) + \sin x\left(\frac{1}{x} - \ln x\right)\right]$$

[6] 6 Find an equation of the tangent line to the curve $2e^{xy} = \sin^{-1} x + y$, at point $(0, 2)$.

Taking the derivative of both sides of the equation, chain rule followed by product rule on the left, gives

$$2e^{xy}\left(y + xy'\right) = \frac{1}{\sqrt{1-x^2}} + y' \Rightarrow 2ye^{xy} + 2xy'e^{xy} = \frac{1}{\sqrt{1-x^2}} + y'$$

Isolating y' gives

$$2xy'e^{xy} - y' = \frac{1}{\sqrt{1-x^2}} - 2ye^{xy} \Rightarrow y'\left(2xe^{xy} - 1\right) = \frac{1 - 2ye^{xy}\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

then solving for y' gives

$$y' = \frac{dy}{dx} = \frac{1 - 2ye^{xy}\sqrt{1-x^2}}{\sqrt{1-x^2}\left(2xe^{xy} - 1\right)}$$

To find the equation of the tangent line we need the slope at $(0, 2)$. Substituting $x = 0$ and $y = 2$ into the expression above for $\frac{dy}{dx}$ gives

$$m = \frac{dy}{dx}\bigg|_{\{} x = 0, y = 2\} = \frac{1 - 2(2)e^0\sqrt{1-0^2}}{\sqrt{1-0^2}\left(2(0)e^0 - 1\right)} = \frac{1-4}{-1} = 3$$

So the slope of the line is $m = 3$ and the y-intercept is given as $(0, 2)$. The equation of the tangent line is $y = 3x + 2$.

[2] 7 (a) Recall that if a function f has an inverse function g, then $g\left(f(x)\right) = x$. Use this fact and chain rule to show that

$$g'\left(f(x)\right) = \frac{1}{f'(x)}$$

By the chain rule, since $\frac{d}{dx}x = 1$,

$$g'\left(f(x)\right)f'(x) = 1 \Rightarrow g'\left(f(x)\right) = \frac{1}{f'(x)}$$

[2] (b) Let $f(x) = 3 + x + \ln x$. Find $f'(x)$ and use the derivative to explain how you know that f has an inverse function.

Using basic differentiation rules gives

$$f'(x) = 1 + \frac{1}{x}$$

Since the domain of f is $(0, \infty)$, i.e., $x > 0$, for all x in the domain of f, $f'(x) > 0$. This means that f is increasing over its domain. Since f is increasing over its domain it is one-to-one, i.e., no horizontal line intersects the graph of f more than once. Thus, f has an inverse function, because any function that is one-to-one has an inverse function.

[1] (c) Let g be the inverse of the function f in part (b) above. Find the value of $g(4)$.

To find $g(4)$ we must solve the equation $f(x) = 3 + x + \ln x = 4$ for x. This gives $x + \ln x = 1 \Rightarrow x = 1$, since $\ln 1 = 0$. Note, since f is one-to-one, we know that this is the only solution.

[2] (d) For the inverse function g in part (c) above find $g'(4)$.

Using the results from parts (a) and (c), we have $f(1) = 4$ and

$$g'(4) = g'\left(f(1)\right) = \frac{1}{f'(1)} = 1$$

8 Consider the piecewise-defined function

$$f(x) = \begin{cases} mx & \text{if } x \le 2 \\ ax^2 + x + 4 & \text{if } x > 2 \end{cases}$$

where m and a are constants.

[2] (a) Find $f'(x)$ as a piecewise-defined function.

Taking the derivative of each piece of f separately gives

$$f(x) = \begin{cases} m & \text{if } x < 2 \\ 2ax + 1 & \text{if } x > 2 \end{cases}$$

Note that we do not know yet whether $f'(2)$ is defined, so $x = 2$ is left out of both pieces.

[4] (b) Find the values of m and a (if any) for which f is both continuous and differentiable at $x = 2$?

If f is continuous at $x = 2$, we must have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \Rightarrow 2m = 4a + 6$$

If f is differentiable at $x = 2$, we must have

$$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x) \Rightarrow m = 4a + 1$$

Subtracting the two equations above gives $m = 5$. Then solving for a, in either equation, gives $a = 1$.

[3] 9 Use the Intermediate Value Theorem to prove that the equation $\cos(x) = x^2$ has at least one solution in the interval $\left[0, \frac{\pi}{2}\right]$.

Let $f(x) = \cos(x) - x^2$. Then

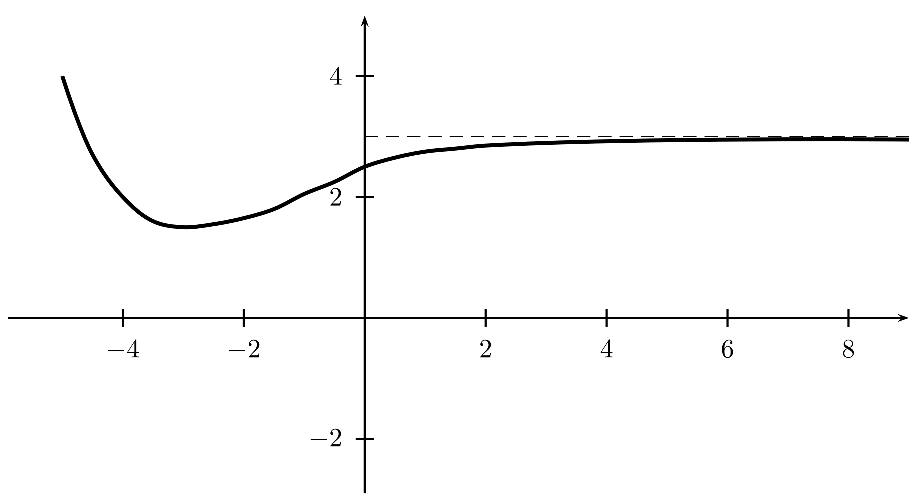
$$f(0) = \cos(0) - 0^2 = 1 > 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^2 = -\frac{\pi^2}{4} < 0$$

By the Intermediate Value Theorem, since $f(0) > 0$ and $f\left(\frac{\pi}{2}\right) < 0$, there is a number c in the open interval $\left(0, \frac{\pi}{2}\right)$ for which $f(c) = 0$. But, if $f(c) = 0$, then $\cos(c) - c^2 = 0 \Rightarrow \cos(c) = c^2$. Hence, c is a solution to the given equation.

10 Consider the function f that is continuous and differentiable on $[-5, \infty)$ which has the following properties:

- $f(-5) = 4$
- $f'(-5) < 0$
- increasing slope for $-5 \le x \le 0$
- local minimum at $x = -3$
- decreasing slope $x > 0$
- horizontal asymptote at $y = 3$, approached as $x \to \infty$

[2] (a) Sketch a possible graph of f using the axes below:



[2] (b) How would you classify the point at $x = 0$?

The point at $x = 0$ is an inflection point of the graph of f. It is a point where the concavity changes from concave up to concave down.

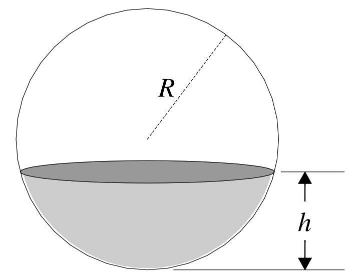
[2] (c) Does f have any local or absolute **maximums**? If so, where?

The function f has no local maximums. The only place where f' changes is sign, i.e., $f'(x) = 0$, is at $x = -3$ where f has a local minimum. There is an absolute maximum at $x = -5$ where $f(-5) = 4$.

11 A spherical segment is formed by cutting a sphere perpendicular to the axis. See the diagram. The volume of a spherical segment of depth h cut from a sphere of radius R is

$$V = \frac{\pi}{3}h^2(3R - h)$$

A hemispherical bowl has a 20 cm radius. It is initially filled with water to a depth of 10 cm. The water leaks out of the a small hole in the bottom of the bowl at a rate of 50 mL/sec.



[3] (a) Find the rate at which the depth of the water in the bowl is decreasing when the water in the bowl is 8 cm deep.

This is a related rates problem with the given rate $\frac{dV}{dt} = -50\,\text{mL/sec} = -50\,\text{cm}^3/\text{sec}$. We want to find $\frac{dh}{dt}$ when $h = 8\,\text{cm}$. The relation between V and h is given, but first write it as

$$V = \frac{\pi}{3}\left(3Rh^2 - h^3\right)$$

Then take the derivative with respect to t remembering that both V and h are functions of t. This gives

$$\frac{dV}{dt} = \frac{\pi}{3}\left(6Rh - 3h^2\right)\frac{dh}{dt} = \pi\left(2Rh - h^2\right)\frac{dh}{dt} = \pi h(2R - h)\frac{dh}{dt}$$

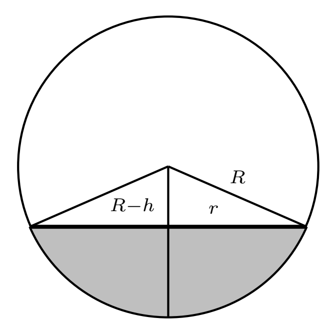
Solving for $\frac{dh}{dt}$ gives

$$\frac{dh}{dt} = \frac{1}{\pi h(2R - h)}\frac{dV}{dt}$$

Now substitute the given rate and the value of h at the given instant:

$$\frac{dh}{dt} = \frac{1}{8\pi(2(20) - 8)}(-50) = -\frac{50}{256\pi}\,\text{cm/sec} = -0.0622\,\text{cm/sec}$$

[5] (b) Use the diagram below to find a relation between the depth h of the water in the tank and radius r of the surface of the water in the tank. Then use this relation and the rate of change found in part (a) above to find the rate at which the area of the surface of the water in the bowl is decreasing when the water in the bowl is 8 cm deep.



From the diagram we have

$$r^2 = R^2 - (R - h)^2 = R^2 - R^2 + 2Rh - h^2 = 2Rh - h^2$$

Thus, the area of the exposed surface of the water in the tank is

$$A = \pi r^2 = \pi\left(2Rh - h^2\right)$$

Then taking the derivative of both side with respect to t, remembering that A and h are functions of t, gives

$$\frac{dA}{dt} = \pi(2R - 2h)\frac{dh}{dt} = 2\pi(R - h)\frac{dh}{dt}$$

Substituting for $\frac{dh}{dt}$ from part (a) above gives

$$\frac{dA}{dt} = \frac{2\pi(R - h)}{\pi h(2R - h)}\frac{dV}{dt} = \frac{2(R - h)}{2R - h}\frac{dV}{dt}$$

Now substitute the given rate and the value of h at the given instant:

$$\frac{dA}{dt} = \left(\frac{2(20 - 8)}{40 - 8}\right)(-50) = -\frac{75}{2}\,\text{cm}^2/\text{sec}$$

12 Let $f(x) = x \ln x$.

[3] (a) Find the local linearization, $L(x)$, for $f(x)$ at $x = 1$.

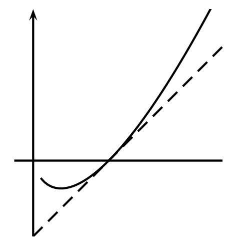
First find derivative:

$$f'(x) = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

Then, $f(1) = 0$ and $f'(1) = 1$, so that the local linearization is

$$L(x) = f(1) + f'(1)(x-1) = x - 1$$

[2] (b) The graph of f is shown. Draw the graph of $L(x)$ on this graph. Does the local linearization give an overestimate or an underestimate?



From the graph we see that the local linearization is an underestimate, since the graph of f is concave up.

[3] (c) Find the third-degree Taylor polynomial

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

of f at $a = 1$. Do not expand the polynomial.

First find the second and third derivatives:

$$f''(x) = \frac{1}{x} \quad \text{and} \quad f'''(x) = -\frac{1}{x^2}$$

Hence, $f''(1) = 1$ and $f'''(1) = -1$. Then the third-degree Taylor polynomial is

$$\begin{aligned} T_3 &= f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3 \\ &= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 \end{aligned}$$

[1] (d) Use the Taylor polynomial from part (c) above to estimate the value of $1.2 \ln 1.2$.

Using the Taylor polynomial from part (c) we have

$$f(1.2) \approx T_3(1.2) = 0.2 + \frac{1}{2}(0.2)^2 - \frac{1}{6}(0.2)^3 = 0.218667$$

The exact value is 0.218786.

13 In forestry the volume of wood in a tree is estimated by measuring the circumference of the tree at the base and computing the radius of the base of the tree knowing the circumference. Then the volume of the tree is computed by assuming that the tree is a cone whose height is approximately 100 times the radius of the base.

[2] (a) Given that the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$, find a formula for the volume estimate for a tree whose circumference at the base is C. Express the volume as function of C.

The circumference of a circle of radius r is $C = 2\pi r$, thus the radius is $r = \dfrac{C}{2\pi}$. Further, if the height of the cone is 100 times the radius, then $h = 100r = \dfrac{50C}{\pi}$. So the volume is

$$V = \frac{\pi}{3}\left(\frac{C}{2\pi}\right)^2\left(\frac{50C}{\pi}\right) = \frac{25C^3}{6\pi^2}$$

[3] (b) The base circumference of a tree is measured to be 80 cm with a maximum error of 1 cm. Use differentials to estimate the maximum possible error in the calculated volume of the tree.

The differential in the volume V is

$$dV = \frac{25}{6\pi^2}\left(3C^2\,dC\right) = \frac{25C}{2\pi^2}\,dC$$

Then, when the circumference is measured to be $C = 80\,\text{cm}$ with a maximum error of $dC = 1\,\text{cm}$, the maximum possible error in the calculated value of the volume is

$$dV = \frac{25\,(80\,\text{cm})^2}{2\pi^2}\,(1\,\text{cm}) = \frac{80000\,\text{cm}^3}{\pi^2} = 8106\,\text{cm}^3$$

[5] 14 Find the point(s) on the curve $y^2 = x^2 + 7$ that are closest to the point $(6, 0)$. Justify your answer.

Use the distance formula to find the distance between the curve and the point. This gives

$$D = \sqrt{(x-6)^2 + y^2} = \sqrt{(x-2)^2 + x^2 + 7} = \sqrt{2x^2 - 4x + 11}$$

To make the calculation of the derivative easier, note that if the square of the distance is minimum the distance is minimum. Thus we minimize

$$D^2 = 2x^2 - 4x + 11$$

Take the derivative to find the critical number(s):

$$\frac{d\,D^2}{dx} = 4x - 4 = 0 \Rightarrow x = 1$$

There is only one critical number at $x = 1$. Check to make sure that this critical number is a minimum. Use the second derivative test.

$$\frac{d^2\,D^2}{dx^2} = 4 > 0 \Rightarrow \quad \text{the critical number is a local minimum}$$

Since there is only one critical number and it is a local minimum, it must be the absolute minimum. Thus, any point on the curve closest to $(6, 0)$ has x-coordinate $x = 1$. Solving for y in the equation for the curve gives

$$y^2 = 1^2 + 7 = 8 \Rightarrow y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Hence, there are two points on the curve closest to $(6, 0)$. They are $\left(1, 2\sqrt{2}\right)$ and $\left(1, -2\sqrt{2}\right)$.

15 Let $f(x) = e^{-2x^2+8x}$. Then

$$f'(x) = 4e^{-2x^2+8x}(2-x)$$
$$f''(x) = 4e^{-2x^2+8x}(2x-3)(2x-5)$$

[2] (a) Find the critical numbers of f.

The critical numbers are the x values for which $f'(x) = 0$. This gives

$$f'(x) = 4e^{-2x^2+8x}(2-x) = 0 \Rightarrow x = 2$$

[4] (b) Find the intervals where f is increasing/decreasing and classify each critical number as a local maximum, minimum, or neither. Does f have an absolute maximum or absolute minimum? If so, where do they occur and what are the absolute extreme values?

For $x < 2$ we have $f'(x) > 0$ and for $x > 0$ we have $f'(x) < 0$. Hence, f is increasing on the interval $(-\infty, 2)$ and is decreasing on the interval $(6, \infty)$. Since f increasing for $x < 2$ and decreasing for $x > 2$, there is a local maximum at $x = 2$. There is not local minimum. Since there is only one local maximum over the domain of the function, it is also the absolute maximum. Thus, there is an absolute maximum at $x = 1$ and no absolute minimum. The absolute maximum value is

$$f(2) = e^{-2\left(2^2\right)+8(2)} = e^8$$

[3] (c) Find the intervals where the graph of f is concave up/down and any inflection points.

The concavity changes where $f''(x) = 0$. This gives

$$f''(x) = 4e^{-2x^2+8x}(2x-3)(2x-5) = 0 \Rightarrow x = \frac{3}{2}, \frac{5}{2}$$

In the interval $\left(-\infty, \frac{3}{2}\right)$, using $x = 0$ as test value, we have $f''(x) > 0$. In the interval $\left(\frac{3}{2}, \frac{5}{2}\right)$, using $x = 2$ as test value, we have $f''(x) < 0$. In the interval $\left(\frac{5}{2}, \infty\right)$, using $x = 3$ as test value, we have $f''(x) > 0$. Thus, the graph of f is concave up on $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$ and is concave down on $\left(\frac{3}{2}, \frac{5}{2}\right)$. There are inflection points at $x = \frac{3}{2}$ and $x = \frac{5}{2}$.