Student Name: ______________________

Student Number: ______________________

Total Marks: ________
100

Okanagan University College
Final Examination

Math 112 (Fall, 2002)

Instructor(s): Clint Lee

Section(s): 71 & 72

December 17, 2002

9:00AM

Duration: 3 hours

READ INSTRUCTIONS CAREFULLY BEFORE COMMENCING EXAM

INSTRUCTIONS: Answer all 15 questions in the spaces provided, showing all significant steps. Partial marks will be awarded for correct work even if the final answer is incorrect. Marks per question are given in the left margin, total 100. Check that your paper contains all 11 pages in addition to the cover page.

This paper contains pages numbered 1 to 11

EXAM BOOKLETS ARE NOT REQUIRED

1 For each of the following, either evaluate the limit or show that the limit does not exist. Show all your calculations and justify all your claims.

[2] (a) $\lim\limits_{x \to 2} \dfrac{x-2}{x^3+3x^2-10x}$

[2] (b) $\lim\limits_{x \to 1} \dfrac{\arcsin x + \arctan x}{x+1}$

[2] (c) $\lim\limits_{x \to 1^-} \dfrac{x}{\ln(x)}$

[2] (d) $\lim\limits_{x \to \infty} \dfrac{\sin(x^2)}{x}$

[4] 2 (a) Given $f(x) = \dfrac{4}{x+2}$, use the **limit definition** of derivative to show that $f'(x) = \dfrac{-4}{(x+2)^2}$.

[2] (b) Use the result of part (a) to find the equation of the tangent line to the graph of $f(x) = \dfrac{4}{x+2}$ at the point where $x = 1$.

3 Suppose that the function f is given by the following table of values.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	2.31	2.36	2.40	2.42	2.50	2.54	2.49

[2] (a) Compute the **average** rate of change of f between $x = 0.0$ and $x = 3.0$.

[2] (b) **Estimate** $f'(1.5)$. Give the best numerical estimate you can.

4 Find the indicated derivative(s) of each function. You may use logarithmic differentiation if you think it will be helpful.

[3] (a) $f(x) = (\sin(\cos 4x))^3$, find $f'(x)$

[4] (b) $y = x \arctan(2x)$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

[4] (c) $p(r) = \sqrt{\dfrac{r^2+1}{r^2-1}}$, find $p'(r)$

[3] (d) Given: $G(t) = f(h(t))$, $h(1) = 4$, $f'(4) = 3$, and $h'(1) = -6$. Find $G'(1)$.

[4] 5 Use logarithmic differentiation to find $f'(x)$ for $f(x) = x^{(\sin x + \cos x)}$.

[4] 6 Find an equation of the tangent line to the curve $2e^{xy} = \sin^{-1} x + y$, at point $(0, 2)$.

[2] 7 (a) Recall that if a function f has an inverse function g, then $g\left(f(x)\right) = x$. Use this fact and chain rule to show that

$$g'\left(f(x)\right) = \frac{1}{f'(x)}$$

[3] (b) Let $f(x) = 3 + x + \ln x$. Find $f'(x)$ and use the derivative to explain how you know that f has an inverse function.

[1] (c) Let g be the inverse of the function f in part (b) above. Find the value of $g(4)$.

[2] (d) For the inverse function g in part (c) above find $g'(4)$.

8 Consider the piecewise-defined function

$$f(x) = \begin{cases} mx & \text{if } x \leq 2 \\ ax^2 + x + 4 & \text{if } x > 2 \end{cases}$$

where m and a are constants.

[2] (a) Find $f'(x)$ as a piecewise-defined function.

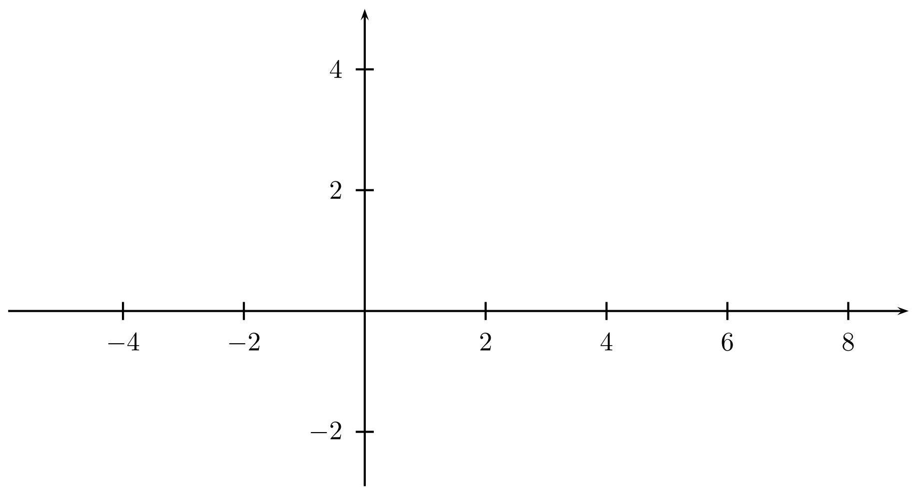
[3] (b) Find the values of m and a (if any) for which f is both continuous and differentiable at $x = 2$?

[3] 9 Use the Intermediate Value Theorem to prove that the equation $\cos(x) = x^2$ has at least one solution in the interval $\left[0, \frac{\pi}{2}\right]$.

10 Consider the function f that is continuous and differentiable on $[-5, \infty)$ which has the following properties:

- $f(-5) = 4$
- $f'(-5) < 0$
- increasing slope for $-5 \leq x \leq 0$
- local minimum at $x = -3$
- decreasing slope $x > 0$
- horizontal asymptote at $y = 3$, approached as $x \to \infty$

[3] (a) Sketch a possible graph of f using the axes below:



[2] (b) How would you classify the point at $x = 0$?

[2] (c) Does f have any local or absolute **maximums**? If so, where?

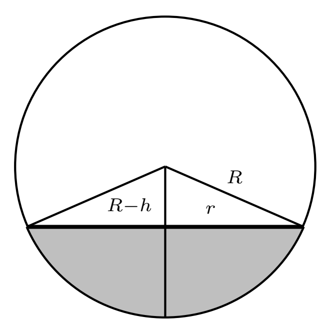
11 A spherical segment is formed by cutting a sphere perpendicular to the axis. See the diagram. The volume of a spherical segment of depth h cut from a sphere of radius R is

$$V = \frac{\pi}{3} h^2 (3R - h)$$

A hemispherical bowl has a 20 cm radius. It is initially filled with water to a depth of 10 cm. The water leaks out of the a small hole in the bottom of the bowl at a rate of 50 mL/sec.

[3] (a) Find the rate at which the depth of the water in the bowl is decreasing when the water in the bowl is 8 cm deep.

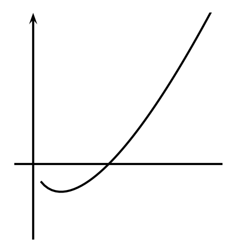
[5] (b) Use the diagram below to find a relation between the depth h of the water in the tank and radius r of the surface of the water in the tank. Then use this relation and the rate of change found in part (a) above to find the rate at which the area of the surface of the water in the bowl is decreasing when the water in the bowl is 8 cm deep.



12 Let $f(x) = x \ln x$.

[3] (a) Find the local linearization, $L(x)$, for $f(x)$ at $x = 1$.

[2] (b) The graph of f is shown. Draw the graph of $L(x)$ on this graph. Does the local linearization give an overestimate or an underestimate?



[3] (c) Find the third-degree Taylor polynomial

$$T_3(x) = f(a) + f'(a)\,(x-a) + \frac{1}{2!}f''(a)\,(x-a)^2 + \frac{1}{3!}f'''(a)\,(x-a)^3$$

of f at $a = 1$.

[1] (d) Use the Taylor polynomial from part (c) above to estimate the value of $1.2 \ln 1.2$.

13 In forestry the volume of wood in a tree is estimated by measuring the circumference of the tree at the base and computing the radius of the base of the tree knowing the circumference. Then the volume of the tree is computed by assuming that the tree is a cone whose height is approximately 100 times the radius of the base.

[2] (a) Given that the volume of a cone of radius r and height h is $V = \frac{1}{3}r^2h$, find a formula for the volume estimate for a tree whose circumference at the base is C. Express the volume as function of C.

[3] (b) The base circumference of a tree is measured to be 80 cm with a maximum error of 1 cm. Use differentials to find the maximum possible error in the calculated volume of the tree.

[6] 14 Find the point(s) on the curve $y^2 = x^2 + 7$ that are closest to the point $(6, 0)$. Justify your answer.

15 Let $f(x) = e^{-2x^2+8x}$. Then

$$f'(x) = 4e^{-2x^2+8x}(2-x)$$
$$f''(x) = 4e^{-2x^2+8x}(2x-3)(2x-5)$$

[2] (a) Find the critical numbers of f.

[4] (b) Find the intervals where f is increasing/decreasing and classify each critical number as a local maximum, minimum, or neither. Does f have an absolute maximum or absolute minimum? If so, where do they occur and what are the absolute extreme values?

[3] (c) Find the intervals where the graph of f is concave up/down and any inflection points.