Example 1

(a) Solve the inequality $\frac{x+7}{x-1} \leq x+2$. Plot the solution on the number line and express the solution using interval notation. Hint: First write the inequality with zero on the right hand side, then get a common denominator on the left. **Do not clear the fraction.**

Solution

Following the hint and writing the inequality with zero on the right hand side gives

$$\frac{x+7}{x-1} \leq x+2 \Rightarrow \frac{x+7}{x-1} - (x+2) \leq 0$$

Then writing over a common denominator gives

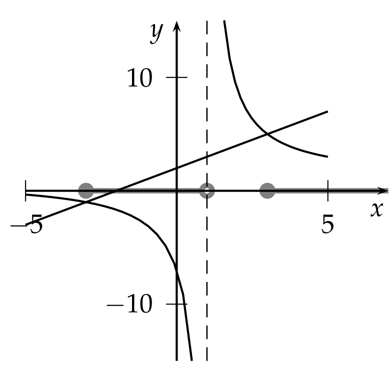
$$\frac{x+7}{x-1} - (x+2) \leq 0 \Rightarrow \frac{x+7-(x-1)(x+2)}{x-1} \leq 0 \Rightarrow \frac{x+7-x^2-x+2}{x-1} \leq 0 \Rightarrow \frac{9-x^2}{x-1} \leq 0$$

The numerator of this expression is zero at $x = \pm 3$ and the denominator is zero at $x = 1$. These are the only values where the expression on the left of the inequality can change sign, and they divide the number line into four intervals: $(-\infty, -3)$, $(-3, 1)$, $(1, 3)$, and $(3, \infty)$. Using $x = -4$, $x = 0$, $x = 2$, and $x = 4$ as test values, we see that the left side of the inequality is negative on the intervals $(-3, 1)$ and $(3, \infty)$. Since equality is allowed, but the expression is not defined for $x = 1$, the solution set is $[-3, 1) \cup [3, \infty)$. Note that we include -3 and 3 since the inequality is less than or equal to, but we must exclude $x = 1$ since the original expression is not defined there. The graph of the solution on the number line is

(b) Draw the graphs of the two functions $f(x) = \frac{x+7}{x-1}$ and $g(x) = x+2$ on the same set of axes. Find the points of intersection of the two graphs and explain how the solution in part (a) relates to the two graphs.

Solution

The plot of the two curves is shown below. The inequality $\frac{x+7}{x-1} \leq x+2$ means that the curve $y = \frac{x+7}{x-1}$ lies below, or just intersects, the line $y = x+2$. From the algebraic solution above we see that the points of intersection of the line and curve are at $x = \pm 3$. Further, from the graph we see that the curve changes from being below the line to being above the line at the vertical asymptote $x = 1$. Thus, the curve is below the line for $-3 \leq x < 1$ and $x \geq 3$. This agrees with the solution in part (a) above.



Example 2

(a) Find the equation of the line L_1 joining the points $A(2,1)$ and $B(4,4)$.

Solution

The slope of the line is

$$m_{AB} = \frac{4-1}{4-2} = \frac{3}{2}$$

Using the point-slope equation gives an equation of the line as

$$y - 1 = \frac{3}{2}(x-2) = \frac{3}{2}x - 3 \Rightarrow y = \frac{3}{2}x - 2$$

(b) Find the equation of the line L_2 through the point $C(1,6)$ that is parallel to the line L_1.

Solution

Two lines are parallel if and only if they have the same slope. So an equation of line L_2 is

$$y - 6 = \frac{3}{2}(x-1) = \frac{3}{2}x - \frac{3}{2} \Rightarrow y = \frac{3}{2}x + \frac{9}{2}$$

(c) Verify that the line joining the points B and C is perpendicular to lines L_1 and L_2. Use this fact to find the distance between lines L_1 and L_2.

Solution

The slope of the line joining points B and C is

$$m_{BC} = \frac{6-4}{1-4} = -\frac{2}{3}$$

Two lines, neither of which is vertical, are perpendicular if the product of their slopes is -1. Here

$$m_{AB} \times m_{BC} = \left(\frac{3}{2}\right)\left(-\frac{2}{3}\right) = -1$$

Hence the line joining points B and C is perpendicular to both L_1 and L_2. Hence, the length of the line segment BC is the distance between lines L_1 and L_2, which is

$$\overline{BC} = \sqrt{(4-1)^2 + (4-6)^2} = \sqrt{13}$$

(d) Find the area of the triangle ABC.

Solution

Since ABC is a right triangle with AB and BC as the perpendicular sides, its area is

$$A_{ABC} = \frac{1}{2}\overline{AB} \cdot \overline{BC}$$

where $\overline{BC} = \sqrt{13}$ and

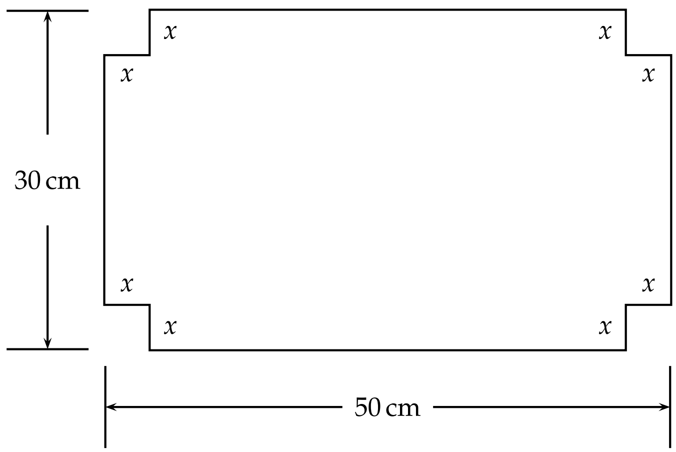
$$\overline{AB} = \sqrt{(4-2)^2 + (4-1)^2} = \sqrt{13}$$

So the area is

$$A_{ABC} = \frac{1}{2}\left(\sqrt{13}\right)^2 = \frac{13}{2}$$

Example 3

(a) A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 30 cm by 50 cm by cutting out equal squares of side length x at each corner and then folding up the sides, as shown. Express the volume V of the box as a function of x.



Solution

The length of the box is $50 - 2x$, the width of the box is $30 - 2x$, and the depth of the box is x. Thus the volume is

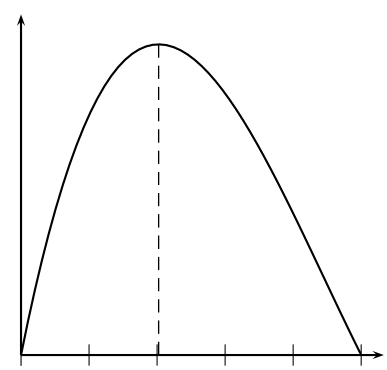
$$V = x(50 - 2x)(30 - 2x) = 4x(25 - x)(15 - x)$$

(b) Give the domain of the function in part (a) that makes sense for the problem stated. Plot the function from part (a) and from your graph estimate the value of x that gives the maximum volume of the box.

Solution

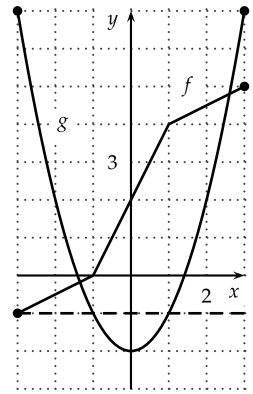
The domain of the function above that makes for this problem is the set of x values for which the volume above is non-negative. Thus, x must be between 0 and 15. Another way to see this is to observe that the squares being cut out cannot take more than the shortest side of the piece of cardboard. Thus, the side length of the square cannot be any bigger than half of the 30 cm side, that is, $0 \leq x \leq 15$.

A graph of the function in part (a) is shown The highest point on the graph appears to be at about $x = 6$. So the maximum volume is achieved when $x \approx 6$ cm.



Example 4

The graphs of the two functions f and g are shown.



(a) Give the domain and range of both f and g.

Solution

The domain of both functions is $[-3, 3]$. The range of f is $[1, 5]$ and the range of g is $[-2, 7]$.

(b) Solve the equation $g(x) = -1$.

Solution

The solution to this equation gives the x-coordinates of the points of intersection of the graph of g with the line $y = -1$. As we see from the graph above, this gives $x = -1$ and $x = 1$.

(c) Given that the graph of g is a parabola, find a formula for the function g.

Solution

The vertex of the parabola is at $(0, -2)$. Thus, $g(x) = ax^2 - 2$ where the constant a can be determined from one point on the graph. Since the graph goes through the point $(3, 7)$ we have $g(3) = 9a - 2 = 7 \Rightarrow a = 1$. So, $g(x) = x^2 - 2$.

(d) Find a formula for the piecewise function f.

Solution

For $-3 \le x \le -1$ the graph of f is a line through points $(-3, -1)$ and $(-1, 0)$ with equation $y = \frac{1}{2}x + \frac{1}{2}$. For $-1 \le x \le 1$ the graph of f is a line through points $(-1, 0)$ and $(1, 4)$ with equation $y = x + 2$. For $1 \le x \le 3$ the graph of f is a line through points $(1, 4)$ and $(3, 5)$ with equation $y = \frac{1}{2}x + \frac{7}{2}$. Thus, the piecewise formula for f is

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{if } -3 \le x \le -1 \\ 2x + 2 & \text{if } -1 < x \le 1 \\ \frac{1}{2}x + \frac{7}{2} & \text{if } 1 < x \le 3 \end{cases}$$