Example 1

Let f be a cubic polynomial function for which

$$f(1) = 6 \text{ and } f(-1) = f(0) = f(2) = 0$$

Determine the value of each of the constants a, b, c, and k for which

$$f(x) = k(x-a)(x-b)(x-c)$$

Expand the expression for the function f in expanded form.

Solution

Since f is a cubic polynomial for which $f(-1) = f(0) = f(2) = 0$ we have

$$f(x) = k(x-(-1))(x-0)(x-2) = kx(x+1)(x-2)$$

Hence, $a = -1$, $b = 0$, and $c = 2$. To find k use the fact that $f(1) = 6$ gives

$$f(1) = k(1)(1+1)(1-2) = -2k = 6 \Rightarrow k = -3$$

Therefore,

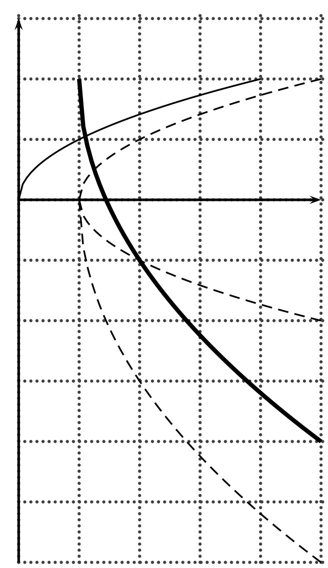
$$f(x) = -3x(x+1)(x-2) = -3x\left(x^2 - x - 2\right) = -3x^3 + 3x^2 + 6x$$

Example 2

The graph of $y = \sqrt{x}$ is shown. Use appropriate transformations to sketch the graph of

$$y = 2 - 3\sqrt{x-1}$$

Describe each transformation and be sure to specify the order in which the transformations are applied.



Solution

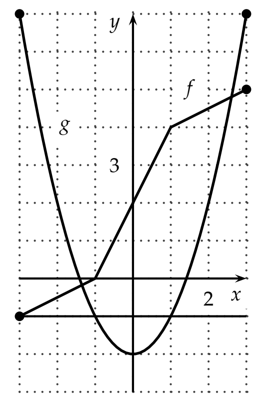
The transformations that produce the graph of $y = 2 - 3\sqrt{x-1}$ from $y = \sqrt{x}$ are

- shift right one unit
- reflect across the x-axis
- stretch vertically by a factor of 3
- shift up 2 units (must be done after the reflection across the x-axis and the vertical stretch)

The graphs of the transformations are shown in the graph above.

Example 3

The graphs of the two functions f and g are shown.



(a) Find $g\left(f(0)\right)$.

Solution

From the graph $f(0) = 2$ and $g(2) = 2$. Hence, $g\left(f(0)\right) = g(2) = 2$.

(b) Find $f\left(g(2)\right)$.

Solution

From the graph $g(2) = 2$ and $f(2) = \frac{5}{2}$. Hence, $f\left(g(2)\right) = f(2) = \frac{9}{2}$.

(c) Find $(f \circ g)(1)$.

Solution

From the graph we have $(f \circ g)(1) = f(g(1)) = f(-1) = 0$.

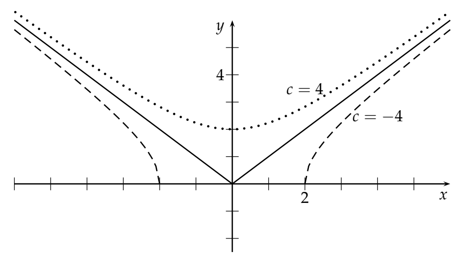
(d) Find $(g \circ f)(-3)$.

Solution

From the graph we have $(g \circ f)(-3) = g(f(-3)) = g(-1) = -1$.

Example 4

The graphs of the function $f(x) = \sqrt{c + x^2}$ for $c = -4$, $c = 0$, and $c = 4$ are shown. Label each graph with the correct value of c, and label both sets of axes to specify the viewing rectangle (window) used.



Solution

For $c = -4$ the function is $f(x) = \sqrt{x^2 - 4}$, so that the domain is $x \leq -2$ or $x \geq 2$. This corresponds to the dashed graph, as labeled. Note that for this graph the x-intercepts are at $x = -2$ and $x = 2$. For $x = 4$ the function is $f(x) = \sqrt{4 + x^2}$, so that $f(0) = 2 \neq 0$. So the graph crosses the y-axis for a positive y value. This corresponds to the dotted graph, as labeled. For $c = 0$ the function is $f(x) = \sqrt{x^2} = |x|$. This corresponds to the unlabeled solid graph. The values given above justify the labeling of the axes shown.

Note that if $y = f(x) = \sqrt{c + x^2}$ then $y^2 = c + x^2 \Rightarrow x^2 - y^2 = -c$. This is the equation of a hyperbola. If $c < 0$, the hyperbola opens along the x-axis. If $c > 0$, the hyperbola opens along the y-axis.