Example 1
Let $k(x) = \dfrac{x^2}{x-1}$.

(a) Write the expression

$$\frac{k(a+h) - k(a)}{h}$$

in terms of a in unsimplified form.

Solution

$$\frac{k(a+h) - k(a)}{h} = \frac{\dfrac{(a+h)^2}{(a+h)-1} - \dfrac{a^2}{a-1}}{h}$$

(b) Simplify the expression in part (a) above to show that

$$\frac{k(a+h) - k(a)}{h} = \frac{a^2 - 2a + ah - h}{(a+h-1)(a-1)}$$

Solution

$$\begin{aligned}
\frac{k(a+h) - k(a)}{h} &= \frac{1}{h}\left[\frac{(a+h)^2}{a+h-1} - \frac{a^2}{a-1}\right] \\
&= \frac{1}{h}\left[\frac{a^2 + 2ah + h^2}{a+h-1} - \frac{a^2}{a-1}\right] \\
&= \frac{1}{h}\left[\frac{\left(a^2 + 2ah + h^2\right)(a-1) - a^2(a+h-1)}{(a-1)(a+h-1)}\right] \\
&= \frac{1}{h}\left[\frac{a^3 + 2a^2h + ah^2 - a^2 - 2ah - h^2 - a^3 - a^2h + a^2}{(a-1)(a+h-1)}\right] \\
&= \frac{1}{h}\left[\frac{a^2h + ah^2 - 2ah - h^2}{(a-1)(a+h-1)}\right] \\
&= \frac{1}{h}\left[\frac{h\left(a^2 + ah - 2a - h\right)}{(a-1)(a+h-1)}\right] \\
&= \frac{a^2 + ah - 2a - h}{(a-1)(a+h-1)}
\end{aligned}$$

Example 1 continued . . .

...Example 1 continued

(c) Find and simplify the expression

$$\frac{k\,(a+h)-k\,(a-h)}{2h}$$

Solution

$$\begin{aligned}
\frac{k\,(a+h)-k\,(a-h)}{2h} &= \frac{1}{2h}\left[\frac{(a+h)^2}{a+h-1}-\frac{(a-h)^2}{a-h-1}\right]\\
&= \frac{1}{2h}\left[\frac{(a+h)^2(a-h-1)-(a-h)^2(a+h-1)}{(a-h-1)(a+h-1)}\right]\\
&= \frac{1}{2h}\left[\frac{(a+h)^2(a-h)-(a+h)^2-(a-h)^2(a+h)+(a-h)^2}{(a-h-1)(a+h-1)}\right]\\
&= \frac{1}{2h}\left[\frac{(a+h)(a-h)(a+h-a+h)-a^2-2ah-h^2+a^2-2ah+h^2}{(a-h-1)(a+h-1)}\right]\\
&= \frac{1}{2h}\left[\frac{2h\left(a^2-h^2\right)-4ah}{(a-h-1)(a+h-1)}\right]\\
&= \frac{a^2-h^2-2a}{(a-h-1)(a+h-1)}
\end{aligned}$$

Example 2

Let $f(x)=\log_2 x$. Show that

$$\frac{f(x+h)-f(x)}{h}=\log_2\left(1+\frac{h}{x}\right)^{1/h}$$

Solution

First write

$$\frac{f(x+h)-f(x)}{h}=\frac{\log_2(x+h)-\log_2 x}{h}=\frac{1}{h}\left(\log_2(x+h)-\log_2 x\right)$$

Using logarithm property $\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$, we have

$$\log_2(x+h)-\log_2 x=\log_2\left(\frac{x+h}{x}\right)=\log_2\left(1+\frac{h}{x}\right)$$

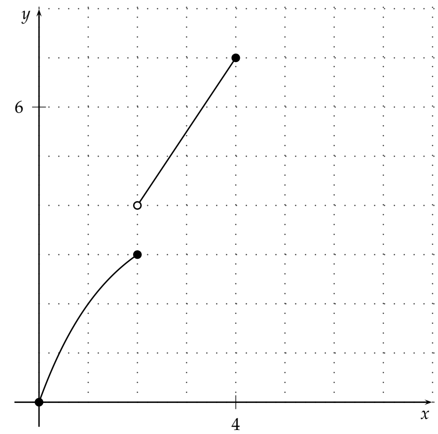
Further, using the logarithm property $log_a x^r = r\log_a x$, we have

$$\frac{1}{h}\left(\log_2(x+h)-\log_2 x\right)=\log_2\left(1+\frac{h}{x}\right)^{1/h}$$

which verifies that

$$\frac{f(x+h)-f(x)}{h}=\log_2\left(1+\frac{h}{x}\right)^{1/h}$$

Example 3



The graph of a function f is shown and a partial formula for f is given below.

$$f(x) = \begin{cases} 4\left(1-2^{-x}\right) & \text{if } 0 \le x < 2 \\ \underline{\hspace{3cm}} & \text{if } 2 \le x \le 4 \end{cases}$$

(a) Give the domain and range of f, expressed as intervals.

Solution

The domain is $[0,4]$ and the range is $[0,3] \cup (4,7]$.

(b) Complete the formula for the function f.

Solution

The upper branch of the function is a straight line through the points (243) and $(4,7)$. The slope of the line is

$$m = \frac{7-4}{4-2} = \frac{3}{2}$$

The equation of the line is

$$y - 4 = \frac{3}{2}(x-2) = \frac{3}{2}x - 3 \Rightarrow y = \frac{3}{2}x + 1$$

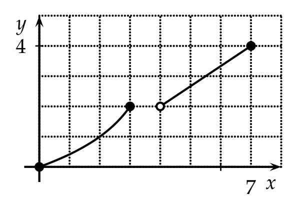
So that

$$f(x) = \frac{3}{2}x + 1 \quad \text{for} \quad 2 \le x \le 4$$

(c) Graph the function $f^{-1}(x)$, the inverse of the function f whose graph is shown above. Give the domain of the inverse.

Solution

The graph of the inverse function $f^{-1}(x)$ is obtained by reflecting the graph of f through the line $y = x$. This is equivalent to interchanging the x and y coordinates in the graph of f. The resulting graph is shown. The domain of the inverse function f^{-1} is the same as the range of the function f, that is, $[0,3] \cup (4,7]$.



(d) Give a formula for f^{-1} by filling in the following piecewise definition:

$$f^{-1}(x) = \begin{cases} -\dfrac{\ln\left(1 - \dfrac{x}{4}\right)}{\ln 2} & \text{if } 0 \le x < 3 \\ \frac{2}{3}(y-1) & \text{if } 4 < x \le 7 \end{cases}$$

Solution

To find the inverse, solve for x in terms of y. To solve for x in upper branch in the piecewise definition first isolate the exponential.

$$y = 4\left(1 - 2^{-x}\right) \Rightarrow \frac{y}{4} = 1 - 2^{-x} \Rightarrow 2^{-x} = 1 - \frac{y}{4}$$

Then take the natural logarithm to solve for x.

$$-x \ln 2 = \ln\left(1 - \frac{y}{4}\right) \Rightarrow x = -\frac{\ln\left(1 - \frac{y}{4}\right)}{\ln 2}$$

The domain of this branch of the inverse is $0 \le y < 3$. For the lower branch we have

$$y = \frac{3}{2}x + 1 \Rightarrow x = \frac{2}{3}(y - 1)$$

The domain of this branch of the inverse is $4 \le y \le 7$. Thus, the complete formula for the inverse is as given above.

Example 4

Simplify the expression: $\tan(\arcsin x)$. Hint: Recall that $\arcsin x = \sin^{-1} x$.

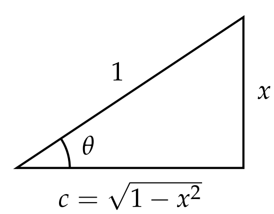
Solution

Let $\theta = \arcsin x$. Then $\sin\theta = x = \dfrac{x}{1}$. In the triangle in the diagram the side opposite the angle θ is x and the hypotenuse is 1. So, by Pythagoras' theorem, the side adjacent to the angle θ is given by

$$c^2 + x^2 = 1^2 \Rightarrow c^2 = 1 - x^2 \Rightarrow c = \sqrt{1 - x^2}$$

Then

$$\tan\theta = \frac{x}{\sqrt{1 - x^2}}$$



Example 5

(a) Use the identity

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

to prove that

$$\cos 3x = \cos x \cos 2x - \sin x \sin 2x$$

Solution

Write $3x = x + 2x$. Then use the given identity with $y = 2x$ to give

$$\cos 3x = \cos(x + 2x) = \cos x \cos 2x - \sin x \sin 2x$$

Example 5 continued ...

...Example 5 continued

(b) Use the identities

$$\sin 2x = 2\sin x\cos x, \quad \cos 2x = \cos^2 x - \sin^2 x, \quad \& \quad \cos^2 x + \sin^2 x = 1$$

to prove that

$$\cos 3x = \cos x\left(1 - 4\sin^2 x\right)$$

Solution

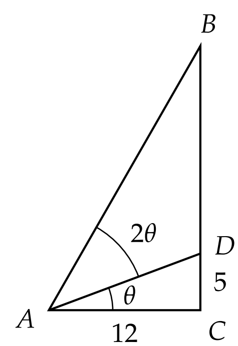
First note that using the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ to replace $\cos^2 x$ with $1 - \sin^2 x$ in the double angle formula for $\cos 2x$ gives

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Then replace $\cos 2x$ by $1 - 2\sin^2 x$ and $\sin 2x$ by $2\sin x\cos x$ in the equation from part (a) to give

$$\cos 3x = \cos x\left(1 - 2\sin^2 x\right) - \sin x\,(2\sin x\cos x) = \cos x\left(1 - 2\sin^2 x - 2\sin^2 x\right) = \cos x\left(1 - 4\sin^2 x\right)$$

(c) In the diagram side $AC = 12$ and side $CD = 5$. Give the value of the angle θ, in degrees, and use the identity in part (b) above to find the length of the segment BD.



Solution

The angle θ is given by

$$\tan\theta = \frac{5}{12} \Rightarrow \theta = \arctan\left(\frac{5}{12}\right) = 22.62^\circ$$

Note that $AD = 13$, so that $\cos\theta = \frac{12}{13}$ and $\sin\theta = \frac{5}{13}$. Then using the identity from part (b) gives

$$\cos 3\theta = \frac{12}{13}\left[1 - 4\left(\frac{5}{13}\right)^2\right] = \frac{828}{2197} = 0.37688$$

Now we can use this value to find AB.

$$\cos 3\theta = \frac{12}{AB} \Rightarrow AB = \frac{12}{\cos 3\theta} = \frac{2197}{69} = 31.84058$$

Then using Pythagoras' Theorem to find BC gives

$$BC = \sqrt{\left(\frac{2197}{69}\right)^2 - 12^2} = \sqrt{\frac{4141225}{4761}} = \frac{2035}{69} = 29.49275$$

Finally

$$BD = BC - 5 = \frac{1690}{69} = 24.49275$$