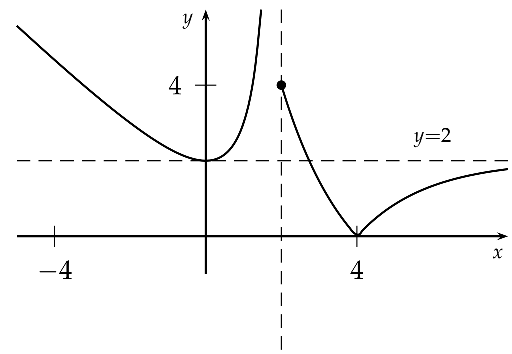
Example 1

The graph of a function f is shown. In parts (a) to (e) compute the limit if it exists. If the limit does not exist, determine if the limit is an infinite limit, and, if it is, state whether it is $+\infty$ or $-\infty$. Otherwise, explain why the limit does not exist.



(a) $\lim_{x\to 4} f(x) = 0$

(b) $\lim_{x\to 2^-} f(x) = \infty$

(c) $\lim_{x\to 2^+} f(x) = 4$

(d) $\lim_{x\to \infty} f(x) = 2$

(e) Give the point(s) where f is discontinuous. Classify any discontinuity as jump, infinite, or removable.

Solution

The function f is discontinuous at $x = 2$ where there is an infinite discontinuity, since one of the one-sided limits is infinite.

Example 2

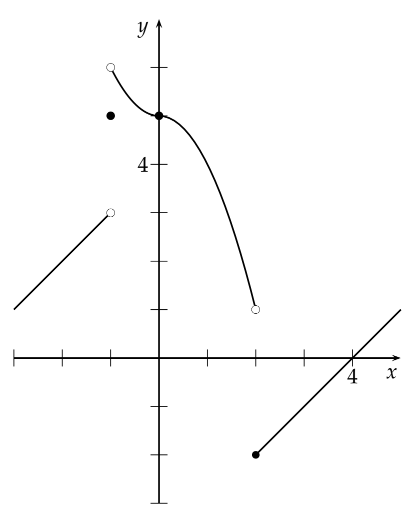
Let

$$Q(x) = \begin{cases} x+4 & \text{for } x < -1 \\ 5 & \text{for } x = -1 \\ x^2+5 & \text{for } -1 < x < 0 \\ 5 - x^2 & \text{for } 0 \le x < 2 \\ x - 4 & \text{for } x \ge 2 \end{cases}$$

(a) Sketch the graph of Q.

Solution

The graph of the function Q is



Example 2 continued . . .

...Example 2 continued

(b) Find the value of each of the following limits, or explain why it does not exist.

(i) $\lim_{x\to -1} Q(x)$ (ii) $\lim_{x\to 0^-} Q(x)$ (iii) $\lim_{x\to 0} Q(x)$ (iv) $\lim_{x\to 2^-} Q(x)$

Solution

From the graph we see that

$$\lim_{x\to -1} Q(x) \text{ DNE, since the two one-sided limits are different}$$
$$\lim_{x\to 0^-} Q(x) = 5$$
$$\lim_{x\to 0} Q(x) = 5$$
$$\lim_{x\to 2^-} Q(x) = 1$$

(c) Which of the following statements is true?

(i) $\lim_{x\to 0} Q(x) = Q(0)$ (ii) $\lim_{x\to -1^+} Q(x) = Q(-1)$

(iii) $\lim_{x\to -1^-} Q(x) = Q(-1)$ (iv) $\lim_{x\to 2^+} Q(x) = Q(2)$

Explain what these limits tell you about the continuity of at $x=-1$, $x=0$, and $x=2$.

Solution

Statement (i) is true since $Q(0) = 5$ and $\lim_{x\to 0} Q(x) = 5$. This shows that Q is continuous at $x = 5$. Statement (ii) is false since $Q(-1) = 5$, but $\lim_{x\to -1^+} Q(x) = 6$. Statement (iii) is false since $Q(-1) = 5$, but $\lim_{x\to -1^-} Q(x) = 4$. Statements (ii) and (iii) together show that Q is discontinuous at $x=-1$. The two one-sided limits are different, so $\lim_{x\to -1} Q(x)$ does not exist. So one of the requirements for continuity fails at $x=-1$. Statement (iv) is true since $Q(2) = -2 = \lim_{x\to 2^+} Q(x)$. This does not guarantee that Q is continuous at $x = 2$. In fact, $\lim_{x\to 2} Q(x)$ does not exist because the two one-sided limits at $x=2$ are different.

Example 3

Find the value of each of the following limits, or explain why it does not exist.

(a) $\lim_{x\to -1} \dfrac{x^2-2x-3}{x^2+2x-3}$

Solution

Direct substitution gives

$$\lim_{x\to -1} \frac{x^2-2x-3}{x^2+2x-3} = \lim_{x\to -1} \frac{(x-3)(x+1)}{(x-1)(x+3)} = \frac{0}{-4} = 0$$

(b) $\lim_{x\to -1} \dfrac{x^2-3x-4}{x^2+3x+2}$

Solution

Direct substitution gives $\frac{0}{0}$, so factor and cancel

$$\lim_{x\to -1} \frac{x^2-3x-4}{x^2+3x+2} = \lim_{x\to -1} \frac{(x-4)(x+1)}{(x+1)(x+2)}$$
$$= \lim_{x\to -1} \frac{x-4}{x+2} = -5$$

Example 3 continued ...

...Example 3 continued

(c) $\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x^2+3x+2}$

Solution

Direct substitution gives $\frac{0}{0}$. First multiply by the root conjugate of the numerator to give

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x^2+3x+2} &= \lim_{x\to -1} \left(\frac{\sqrt{x+5}-2}{x^2+3x+2}\right)\left(\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2}\right) \\
&= \lim_{x\to -1} \frac{x+5-4}{(x+2)(x+1)\left(\sqrt{x+5}+2\right)} \\
&= \lim_{x\to -1} \frac{x+1}{(x+2)(x+1)\left(\sqrt{x+5}+2\right)} \\
&= \lim_{x\to -1} \frac{1}{(x+2)\left(\sqrt{x+5}+2\right)} = \frac{1}{4}
\end{aligned}$$

Example 4

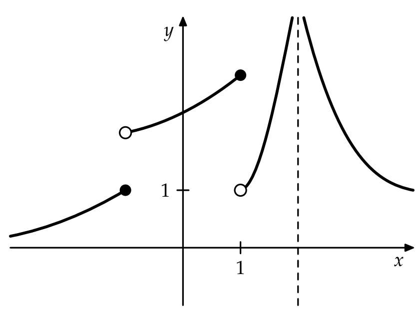
Suppose that you are given a function k for which

$$\lim_{x\to -1^-} k(x) = 1,\ \lim_{x\to -1^+} k(x) = 2,\ \lim_{x\to 1^-} k(x) = 3$$
$$\lim_{x\to 1^+} k(x) = 1,\ \lim_{x\to 2^-} k(x) = \infty,\ \lim_{x\to 2^+} k(x) = \infty$$
$$k(-1) = 1,\ k(1) = 3$$

Sketch a possible graph of k.

Solution

A possible graph looks like this



Example 5

Evaluate each of the following limits and explain what the limit means in terms of a horizontal asymptote for the graph of the function.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{4x^2 + 5x + 3}$ (b) $\displaystyle\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}}$

Solution

(a) This is a rational function with the degree in the numerator and denominator equal. The value of the limit is equal to the ratio of the coefficients of the terms with the highest powers. Thus

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{4x^2 + 5x + 3} = \frac{3}{4}$$

This means that there is a horizontal asymptote along the line $y = \frac{3}{4}$. Since this is a rational function, this is the horizontal asymptote on both the left and the right.

(b) Factoring out the highest power in the denominator gives

$$\frac{x+1}{\sqrt{x^2+1}} = \frac{x\left(1 + \dfrac{1}{x}\right)}{|x|\sqrt{1 + \dfrac{1}{x^2}}}$$

Since $x \to -\infty$, we have $\dfrac{x}{|x|} = -1$. Hence

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}} = -\lim_{x\to-\infty} \frac{\left(1 + \dfrac{1}{x}\right)}{\sqrt{1 + \dfrac{1}{x^2}}} = -1$$

This means that there is a horizontal asymptote along the line $y = -1$ on the left. Since this is not a rational function, we do not know about a possible horizontal asymptote on the right.

Example 6

Find the horizontal and vertical asymptotes for the function

$$f(x) = \frac{2x^2 + 3x - 2}{x^2 + x - 12}$$

Check the answer by graphing the curve and estimating the asymptotes.

Solution

The denominator is $x^2 + x - 12 = (x-3)(x+4)$. This is zero when $x = 3$ or $x = -4$. These are the two vertical asymptotes of the graph of the function f. The behaviour of the function on either side of the two vertical asymptotes is given by

$$\lim_{x \to -4^-} \frac{2x^2 + 3x - 2}{x^2 + x - 12} = +\infty \qquad \lim_{x \to -4^+} \frac{2x^2 + 3x - 2}{x^2 + x - 12} = -\infty$$

$$\lim_{x \to 3^-} \frac{2x^2 + 3x - 2}{x^2 + x - 12} = -\infty \qquad \lim_{x \to 3^+} \frac{2x^2 + 3x - 2}{x^2 + x - 12} = +\infty$$

Further, note that this is a rational function with the highest power in numerator equal to the highest power in the denominator. Thus,

$$\lim_{x \to \pm\infty} \frac{2x^2 + 3x - 2}{x^2 + x - 12} = 2$$

So the graph of the f has a horizontal asymptote along the line $y = 1$. The graph of the function looks like this