

Example 1

Let

$$h(x) = \sqrt{x} + x$$

- (a) Use the definition of the derivative at a point

$$h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$$

to find $h'(4)$.**Solution**

Using the definition gives

$$h'(4) = \lim_{x \rightarrow 4} \frac{h(x) - h(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} + x - 6}{x - 4}$$

Multiplying numerator and denominator by $x - 6 - \sqrt{x}$ gives

$$\begin{aligned} h'(4) &= \lim_{x \rightarrow 4} \left(\frac{\sqrt{x} + x - 6}{x - 4} \right) \left(\frac{x - 6 - \sqrt{x}}{x - 6 - \sqrt{x}} \right) \\ &= \lim_{x \rightarrow 4} \frac{(x - 6)^2 - x}{(x - 4)(x - 6 - \sqrt{x})} = \lim_{x \rightarrow 4} \frac{x^2 - 12x + 36 - x}{(x - 4)(x - 6 - \sqrt{x})} \\ &= \lim_{x \rightarrow 4} \frac{x^2 - 13x + 36}{(x - 4)(x - 6 - \sqrt{x})} = \lim_{x \rightarrow 4} \frac{(x - 4)(x - 9)}{(x - 4)(x - 6 - \sqrt{x})} \\ &= \lim_{x \rightarrow 4} \frac{x - 9}{x - 6 - \sqrt{x}} = \frac{4 - 9}{4 - 6 - 2} = \frac{5}{4} \end{aligned}$$

Another approach is to make the change of variables $u = \sqrt{x} \Rightarrow x = u^2$. This gives

$$\begin{aligned} h'(4) &= \lim_{u \rightarrow 2} \frac{u + u^2 - 6}{u^2 - 4} = \lim_{u \rightarrow 2} \frac{(u - 2)(u + 3)}{(u - 2)(u + 2)} \\ &= \lim_{u \rightarrow 2} \frac{u + 3}{u + 2} = \frac{5}{4} \end{aligned}$$

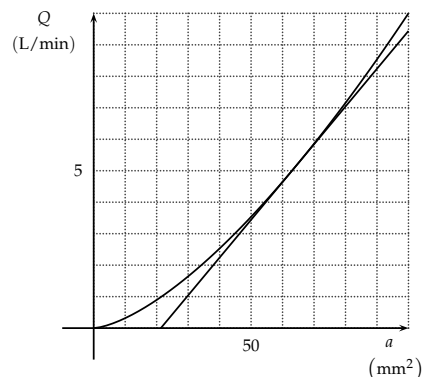
- (b) Use your result in part (a) above to find the equation of the tangent to the curve
- $y = \sqrt{x} + x$
- at the point on the curve where
- $x = 4$
- .

SolutionSince $h(4) = 6$, the tangent line goes through the point $(4, 6)$. So the equation is

$$y - 6 = \frac{5}{4}(x - 4) = \frac{5}{4}x - 5 \Rightarrow y = \frac{5}{4}x + 1$$

Example 2

The function $Q = F(a)$ gives the flow rate, in litres per minute, out of a hole in the bottom of a jug as a function of the area a of the hole in square millimetres. The graph of the function F is shown for a from 0 to 100 mm^2 together with the line tangent to the graph at the point where $a = 64 \text{ mm}^2$.



- (a) Describe what the derivative $\frac{dQ}{da}$ represents and give its units. It is not sufficient to simply say that it is the slope of the graph at a certain point.

Solution

The derivative gives the rate of change of the flow rate with respect to the area of the hole.

- (b) By estimating the slope of the tangent line estimate the value of $F'(64)$.

Solution

From the graph the tangent line goes through the points $(21, 0)$ and $(100, 9.3)$. Hence the slope, and so the value of $F'(64)$, is approximately

$$F'(64) = \left. \frac{dQ}{da} \right|_{a=64} \approx \frac{9.3 - 0}{100 - 21} = 0.117 \frac{\text{L/min}}{\text{mm}^2}$$

- (c) Explain in layman's terms what the value of the derivative in part (b) above tells you.

Solution

If the area of the hole increases from by 1 mm^2 from $a = 64 \text{ mm}^2$ the flow rate will increase by 0.117 L/min.

Example 3

Let

$$F(x) = \frac{3x}{1 + 2x}$$

Use the definition of the derivative

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

to find $F'(x)$.

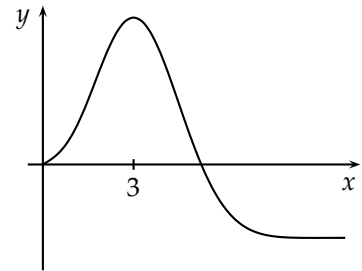
Solution

Using the definition gives

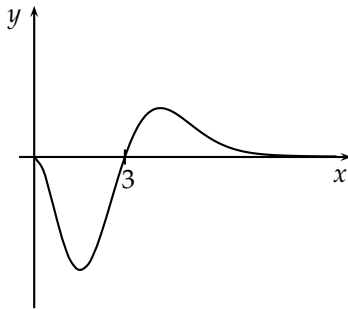
$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{\frac{3(x+h)}{1+2(x+h)} - \frac{3x}{1+2x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3(x+h)}{1+2(x+h)} - \frac{3x}{1+2x} \right) \\ &= \lim_{h \rightarrow 0} \frac{3}{h} \left(\frac{(x+h)(1+2x) - x(1+2x+2h)}{(1+2x+2h)(1+2x)} \right) \\ &= \lim_{h \rightarrow 0} \frac{3}{h} \left(\frac{x + 2x^2 + h + 2xh - x - 2x^2 - 2xh}{(1+2x+2h)(1+2x)} \right) \\ &= \lim_{h \rightarrow 0} \frac{3}{h} \left(\frac{h}{(1+2x+2h)(1+2x)} \right) = \lim_{h \rightarrow 0} \frac{3}{(1+2x+2h)(1+2x)} = \frac{3}{(1+2x)^2} \end{aligned}$$

Example 4

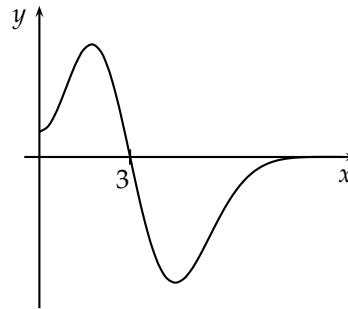
The graph of a function f is shown.



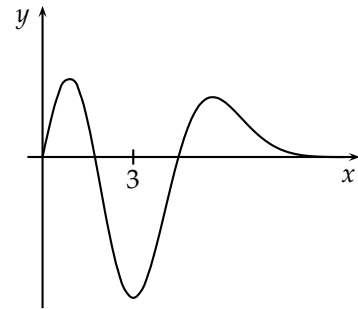
- (a) The graphs of three functions are shown below. Choose the one that is the graph of the derivative of the function f .



(A)



(B)



(C)

Solution

Graph (B) is the graph of the derivative

- (b) Choose the graph from the graphs in part (a) above that is the graph of the second derivative of the function f .

Solution

Graph (C) is the graph of the second derivative.

- (c) Explain your choices in parts (a) and (b).

Solution

The function f is increasing, graph has positive slope, as x increases from zero and has a high point, with zero slope, at $x = 3$. Graph (B) shows positive values for x near zero and a value of zero at $x = 3$. Hence, it must be the graph of the derivative of f . Graph (B) has zero slope at $x = 0$, then positive slope until the high point in the graph to the left of $x = 3$, then negative slope through $x = 3$ until the low point on the graph to the right of $x = 3$. Hence, graph (B) is the graph of the derivative.

Graph (C) shows positive values up to a point to the left of $x = 3$, negative values through $x = 3$, going back to positive values at a point to the right of $x = 3$. Hence, graph (C) is the graph of the derivative of the function in graph (B) and so the graph of the second derivative of the function f .