
Example 1

Find $\dfrac{dy}{dx}$ for $y = 8\sqrt[4]{x^3} - \dfrac{7}{\sqrt{x^3}}$.

Solution

Write the function as $y = 8x^{3/4} - 7x^{-3/2}$. Then use the Power Rule on each term separately to give

$$\frac{dy}{dx} = 8\left(\tfrac{3}{4}\right)x^{3/4-1} - 7\left(-\tfrac{3}{2}\right)x^{-3/2-1} = 6x^{-1/4} + \tfrac{21}{2}x^{-5/2} = \frac{6}{\sqrt[4]{x}} - \frac{21}{2\sqrt{x^5}}$$

Example 2

Find $g'(x)$ and $g''(x)$ for $g(x) = x^2(x+3)e^x$.

Solution

First multiply through by x^2 to give $g(x) = \left(x^3 + 3x^2\right)e^x$. Then use the Product Rule to give the first derivative as

$$g'(x) = \left(3x^2 + 6x\right)e^x + \left(x^3 + 3x^2\right)e^x = \left(x^3 + 6x^2 + 6x\right)e^x = x\left(x^2 + 6x + 6\right)e^x$$

For the second derivative use the second last expression above. Then the Product Rule gives

$$g''(x) = \left(3x^2 + 12x + 6\right)e^x + \left(x^3 + 6x^2 + 6x\right)e^x = \left(x^3 + 9x^2 + 18x + 6\right)e^x$$

Example 3

Find $h'(x)$ for $h(x) = \dfrac{x^2e^x}{x^2+1}$.

Solution

Use the Quotient Rule to obtain

$$h'(x) = \frac{\frac{d}{dx}\left(x^2e^x\right)\left(x^2+1\right) - x^2e^x\,(2x)}{\left(x^2+1\right)^2}$$

Then use the Product Rule for the derivative in the first term in the numerator to obtain

$$\begin{aligned} h'(x) &= \frac{\left(2xe^x + x^2e^x\right)\left(x^2+1\right) - 2x^3e^x}{\left(x^2+1\right)^2} \\ &= \frac{e^x\left(2x^3 + 2x + x^4 + x^2 - 2x^3\right)}{\left(x^2+1\right)^2} \\ &= \frac{x\left(x^3 + x + 2\right)e^x}{\left(x^2+1\right)^2} = \frac{x(x+1)\left(x^2 - x + 2\right)e^x}{\left(x^2+1\right)^2} \end{aligned}$$

Example 4

Suppose that $f(3) = 5$, $g(3) = -2$, $f'(3) = -4$, and $g'(3) = 6$. Let $k(x) = f(x)g(x)$. Find $k'(3)$.

Solution

First use the Product Rule to give

$$k'(x) = f'(x)g(x) + f(x)g'(x)$$

Then substitute $x = 3$ and use the given values to obtain

$$k'(x) = f'(3)g(3) + f(3)g'(3) = (-4)(-2) + 5(6) = 38$$

Example 5

A bookstore currently makes an average profit of \$3 per book, but this is declining by about \$0.05 every month. The bookstore sells about 2000 books every month and this is increasing at the rate of about 40 books every month. Use the Product Rule to estimate the rate at which the bookstore's total monthly profit is either increasing or decreasing, in dollars per month. Explain what each of the two terms in the Product Rule represent in this problem.

Solution

Let $B(t)$ be profit from the sale of each book at time t, in months and $S(t)$ be the number of books sold in month at time t. Let $t = 0$ be the current month, so that the given information can be written as

$$B(0) = \$3/\text{book},\ B'(0) = -\$0.05/\text{book}/\text{month},\ S(0) = 2000\,\text{books},\ S'(0) = 40\,\text{books}/\text{month}$$

If $P(t)$ is the total monthly profit at time t, then $P(t) = B(t)S(t)$. Using the Product Rule

$$P'(t) = B'(t)S(t) + B(t)S'(t)$$

So that for the current month the rate of change of the profit with respect to the time t in months is

$$\begin{aligned} P'(0) &= B'(0)S(0) + B(0)S'(0) = (-\$0.05/\text{book}/\text{month})\,(2000\,\text{books}) + (\$3/\text{book})\,(40\,\text{books}/\text{month}) \\ &= -\$100/\text{month} + \$120/\text{month} = \$20/\text{month} \end{aligned}$$

The total monthly profit is increasing by about \$20 per month. The first term above is the result in the decline in total profit due to the decline in the profit per book and the second term is due to the increase in profit due to increased sales.

Example 6

Find the equation of the tangent line to the curve $y = (x+1)\,e^{-x/2}$ at the point where $x = 0$.

Solution

To find the slope of the tangent, find the value of $\left.\dfrac{dy}{dx}\right|_{x=0}$. Using the Product Rule and the Exponential Function Rule gives

$$\frac{dy}{dx} = (1)e^{-x/2} + (x+1)\,e^{-x/2}\left(-\tfrac{1}{2}\right) = \left(1 - \tfrac{1}{2} - \tfrac{1}{2}x\right)e^{-x/2} = \tfrac{1}{2}\,(1-x)\,e^{-x/2}$$

So the slope of the tangent line where $x = 0$ is

$$m_T = \left.\frac{dy}{dx}\right|_{x=0} = \tfrac{1}{2}\,(1-0)\,e^{-0/2} = \tfrac{1}{2}$$

When $x = 0$ the y-coordinate is $y = (0+1)\,e^{-0/2} = 1$. So the equation of the tangent line at the point $(0,1)$ is

$$y - 1 = \tfrac{1}{2}(x - 0) \Rightarrow y = \tfrac{1}{2}x + 1$$

Example 7

Find the point(s) on the curve $y = \dfrac{x+1}{x}$ where the tangent line goes through the point $(1,1)$.

Solution

First note that the point $(1,1)$ does not lie on the curve, since when $x = 1$, the corresponding y-coordinate on the curve is $y = 2$. Let (a,b) be a point on the curve $y = \dfrac{x+1}{x} = 1 + \dfrac{1}{x}$ where the tangent line goes through the point $(1,1)$. Then the slope of the line through the points (a,b) and $(1,1)$ is

$$m = \frac{b-1}{a-1}$$

Since the point (a,b) lies on the curve, $b = 1 + \dfrac{1}{a}$. So that the slope of the line is

$$m = \frac{\left(1 + \frac{1}{a}\right) - 1}{a-1} = \frac{\frac{1}{a}}{a-1} = \frac{1}{a(a-1)}$$

But, if the line through the points (a,b) and $(1,1)$ is tangent to the curve, its slope is given by the derivative at $x = a$. Here

$$y = 1 + \frac{1}{x} = 1 + x^{-1} \Rightarrow \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$$

So the slope of the line through the points (a,b) and $(1,1)$ is also given by

$$m = \left.\frac{dy}{dx}\right|_{x=a} = -\frac{1}{a^2}$$

Hence we have the equation

$$\frac{1}{a(a-1)} = -\frac{1}{a^2} \Rightarrow a^2 = -a(a-1) = -a^2 + a \Rightarrow a\,(2a-1) = 0$$

Two solutions are $a = 0$ and $a = \frac{1}{2}$. There is a vertical asymptote at $x = 0$, so $a = 0$ is not a valid solution. Therefore, there is only one point on the curve where the tangent line goes through the point $(1,1)$. The y-coordinate of the point is

$$b = 1 + \frac{1}{1/2} = 3$$

so the point is $\left(\frac{1}{2}, 3\right)$.

Example 8

Differentiate and simplify

(a) $w = \left(u^3 - 2\right)^2 \left(u^2 + 3u + 1\right)^4$

Solution

Using the Product Rule followed by the Chain Rule (General Power Rule) gives

$$\begin{aligned}
\frac{dw}{du} &= 2\left(u^3-2\right)\left(3u^2\right)\left(u^2+3u+1\right)^4 + \left(u^3-2\right)^2\left[4\left(u^2+3u+1\right)^3(2u+3)\right] \\
&= \left(u^3-2\right)\left(u^2+3u+1\right)^3\left[6u^2\left(u^2+3u+1\right) + 4\left(u^3-2\right)(2u+3)\right] \\
&= \left(u^3-2\right)\left(u^2+3u+1\right)^3\left(6u^4+18u^3+6u^2+8u^4+12u^3-16u-24\right) \\
&= 2\left(u^3-2\right)\left(u^2+3u+1\right)^3\left(7u^4+15u^3+3u^2-8u-12\right)
\end{aligned}$$

Example 8 continued ...

. . . Example 8 continued

(b) $f(x) = \dfrac{(x+1)^2}{\sqrt{3x-4}}$

Solution

Write the function as

$$f(x) = \frac{(x+1)^2}{(3x-4)^{1/2}}$$

Then using the Quotient Rule

$$\begin{aligned}
f'(x) &= \frac{2(x+1)(3x-4)^{1/2} - (x+1)^2\left(\frac{1}{2}\right)(3x-4)^{-1/2}(3)}{3x-4} \\
&= \frac{(x+1)(3x-4)^{-1/2}[4(3x-4) - 3(x+1)]}{2(3x-4)} \\
&= \frac{(x+1)[4(3x-4) - 3(x+1)]}{2(3x-4)^{3/2}} = \frac{(x+1)(12x-16-3x-3)}{2(3x-4)^{3/2}} \\
&= \frac{(x+1)(9x-19)}{2(3x-4)^{3/2}}
\end{aligned}$$

Alternatively write the function as

$$f(x) = (x+1)^2(3x-4)^{-1/2}$$

Then using the Product Rule followed by the Chain Rule (General Power Rule) gives

$$\begin{aligned}
f'(x) &= 2(x+1)(3x-4)^{-1/2} + (x+1)^2\left(-\frac{1}{2}\right)(3x-4)^{-3/2}(3) \\
&= \frac{2(x+1)}{(3x-4)^{1/2}} - \frac{3(x+1)^2}{2(3x-4)^{3/2}} = \frac{4(x+1)(3x-4) - 3(x+1)^2}{2(3x-4)^{3/2}} \\
&= \frac{(x+1)[4(3x-4) - 3(x+1)]}{2(3x-4)^{3/2}} = \frac{(x+1)(12x-16-3x-3)}{2(3x-4)^{3/2}} \\
&= \frac{(x+1)(9x-19)}{2(3x-4)^{3/2}}
\end{aligned}$$

Example 8 continued . . .

...Example 8 continued

(c) $y = \sqrt{1 + \left(\sqrt[3]{1 + e^{3x}}\right)^2}$

Solution

Write the function as

$$y = \left[1 + \left(1 + e^{3x}\right)^{2/3}\right]^{1/2}$$

Then using the Chain Rule gives

$$\begin{aligned}
\frac{dy}{dx} &= \tfrac{1}{2}\left[1 + \left(1 + e^{3x}\right)^{2/3}\right]^{-1/2} \left(\tfrac{2}{3}\right) \left(1 + e^{3x}\right)^{-1/3} \left(3e^{3x}\right) \\
&= \frac{e^{3x}}{\left[1 + (1 + e^{3x})^{2/3}\right]^{1/2} (1 + e^{3x})^{1/3}} \\
&= \frac{e^{3x}}{\sqrt{1 + (1 + e^{3x})^{2/3}}\,\sqrt[3]{1 + e^{3x}}}
\end{aligned}$$