Example 1

Let $y = \dfrac{\left(x^3-8\right)^3}{\left(x^3+8\right)^2}$.

(a) Use the properties of logarithms to simplify $\ln y$.

Solution

$$\ln y = 3\ln\left(x^3-8\right) - 2\ln\left(x^3+8\right)$$

(b) Use implicit differentiation on the simplified expression for $\ln y$ from part (a) to find $\dfrac{dy}{dx}$. Express the final result in terms of x alone.

Solution

Taking the derivative of $\ln y$ gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{y'}{y}$. Taking derivative of both sides gives

$$\frac{y'}{y} = \frac{3\left(3x^2\right)}{x^3-8} - \frac{2\left(3x^2\right)}{x^3+8} = \frac{3x^2\left(3x^3+24-2x^3+16\right)}{\left(x^3-8\right)\left(x^3+8\right)} = \frac{3x^2\left(x^3+40\right)}{\left(x^3-8\right)\left(x^3+8\right)}$$

Solving for y' and plugging the original expression for y gives

$$\frac{dy}{dx} = y' = y\left(\frac{3x^2\left(x^3+40\right)}{\left(x^3-8\right)\left(x^3+8\right)}\right) = \left(\frac{\left(x^3-8\right)^3}{\left(x^3+8\right)^2}\right)\left(\frac{3x^2\left(x^3+40\right)}{\left(x^3-8\right)\left(x^3+8\right)}\right) = \frac{3x^2\left(x^3-8\right)^2\left(x^3+40\right)}{\left(x^3+8\right)^3}$$

This is the technique of **implicit differentiation**.

Example 2

Use logarithmic differentiation to find the derivative of each of following functions:

(a) $y = xe^{\sin x}$

Solution

Take ln of both sides of equation

$$\ln y = \ln x + \sin x$$

Then take derivative

$$\frac{y'}{y} = \frac{1}{x} + \cos x$$

Multiply by $y = xe^{\sin x}$

$$\frac{dy}{dx} = y' = xe^{\sin x}\left(\frac{1}{x} + \cos x\right) = e^{\sin x} + x\cos x e^{\sin x} = e^{\sin x}\left(1 + x\cos x\right)$$

Example 2 continued . . .

...Example 2 continued

(b) $y = x^{\cos x}$

Solution

Take ln of both sides of equation

$$\ln y = \cos x \ln x$$

Then take derivative

$$\frac{y'}{y} = -\sin x \ln x + \cos x \left(\frac{1}{x}\right) = -\sin x \ln x + \frac{\cos x}{x}$$

Multiply by $y = x^{\cos x}$

$$\frac{dy}{dx} = y' = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right) = x^{\cos x}\left(\frac{-x \sin x \ln x + \cos x}{x}\right)$$

Example 3

Find the point(s) on the curve $y = x^x$ where the tangent line is horizontal.

Solution

Use logarithmic differentiation to find $\frac{dy}{dx}$ in order to find the slope. This gives

$$\ln y = x \ln x$$

Then take derivative

$$\frac{y'}{y} = \ln x + 1$$

Multiply by $y = x^x$

$$\frac{dy}{dx} = y' = x^x(\ln x + 1)$$

The tangent line is horizontal when the slope is zero. Since $x > 0$, this gives

$$\ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$$

The y-coordinate of the point is

$$y = \left(\frac{1}{e}\right)^{1/e} = \frac{1}{e^{(1/e)}}$$

So the point is $\left(\frac{1}{e}, \frac{1}{e^{(1/e)}}\right) = (0.367879, 0.692201)$.

Example 4

When fluid flows through a pipe the radius r of the pipe can be determined by measuring the flow rate R. This could be useful in determining the radii of capillary blood vessels. Under certain conditions the radius is given in terms of the flow rate by the formula

$$r = \left(\frac{R}{25}\right)^{1/4}$$

where R is measured in cm^3/s and r is measured in centimetres.

(a) Calculate the average rate of change of r with respect to the flow rate over the intervals $300 \le R \le 400$ and $400 \le R \le 500$.

Solution

For $300 \le R \le 400$ the average rate of change is

$$\frac{r(400) - r(300)}{400 - 300} = \frac{1}{100}\left[\left(\frac{400}{25}\right)^{1/4} - \left(\frac{300}{25}\right)^{1/4}\right] = 0.001388\,\mathrm{cm}/\left(\mathrm{cm}^3/\mathrm{s}\right)$$

For $400 \le R \le 500$ the average rate of change is

$$\frac{r(500) - r(400)}{500 - 400} = \frac{1}{100}\left[\left(\frac{500}{25}\right)^{1/4} - \left(\frac{400}{25}\right)^{1/4}\right] = 0.001147\,\mathrm{cm}/\left(\mathrm{cm}^3/\mathrm{s}\right)$$

(b) Find the instantaneous rate of change of the radius r of the pipe with respect to flow rate in when the flow rate is $R = 400\,\mathrm{cm}^3/\mathrm{s}$. Give the units of the rate of change and explain in practical terms what it means.

Solution

The instantaneous rate of change is given by the derivative which is

$$\frac{dr}{dR} = \frac{1}{4 \cdot 25^{1/4}} R^{-3/4} = \frac{1}{4\left(25R^3\right)^{1/4}}$$

Evaluating at $R = 400$ gives

$$\left.\frac{dr}{dR}\right|_{R=400} = \frac{1}{800} = 0.00125\,\mathrm{cm}/\left(\mathrm{cm}^3/\mathrm{s}\right)$$

This means that when the flow increases by one unit from $400\,\mathrm{cm}^3/\mathrm{s}$ the radius of the pipe increases by about 0.00125 cm.

Example 5

The mass of a 5-metre tapered beam varies in such a way that the total mass in kilograms between the thick end and a point x metres from the end is

$$m(x) = 70 - \frac{140}{x+2} \quad \text{for} \quad 0 \le x \le 5$$

This is the **mass function** of the beam.

(a) What is the total mass of the beam?

Solution

The total mass of the beam is $m(5) = 70 - 20 = 50\,\mathrm{kg}$.

Example 5 continued ...

...Example 5 continued

(b) The rate of change of the total mass function with respect to the distance for the thick end gives the **linear mass density**. Find the linear mass density of the beam as a function of x.

Solution

The rate of change of the mass function is

$$\frac{dm}{dx} = \frac{140}{(x+2)^2}$$

(c) Find the linear mass density at

i) the thick end of the beam

Solution

At the thick end of the beam $x = 0$. The linear mass density here is

$$\left.\frac{dm}{dx}\right|_{x=0} = \frac{140}{4} = 35\,\text{kg/m}$$

ii) the thin end of the beam

Solution

At the thin end of the beam $x = 5$. The linear mass density here is

$$\left.\frac{dm}{dx}\right|_{x=5} = \frac{140}{49} = 2.857\,\text{kg/m}$$

iii) the middle of the beam

Solution

At the middle of the beam $x = 2.5$. The linear mass density here is

$$\left.\frac{dm}{dx}\right|_{x=2.5} = \frac{140}{4.5^2} = 6.914\,\text{kg/m}$$

(d) What is the average density of the beam? Is there any point on the beam where the linear mass density equals this average density?

Solution

The average density of the beam is

$$\frac{m(5) - m(0)}{5 - 0} = \frac{50}{5} = 10\,\text{kg/m}$$

We need to find an x for which

$$\frac{dm}{dx} = \frac{140}{(x+2)^2} = 10 \Rightarrow (x+2)^2 = 14 \Rightarrow x = \sqrt{14} - 2$$

This is at $x = 1.742\,\text{m}$.

Example 6

A fifty litre tank contains twenty litres of water that has 10 grams of a certain dye dissolved in it. Water containing 1 g/L of the dye is pumped into the tank at 3 L/min and the well mixed dye solution is pumped out at 2 L/min. The total number of grams of dye in the tank after t minutes is given by

$$A(t) = \frac{(20+t)^3 - 4000}{(20+t)^2}$$

(a) Find the rate of change of the number of grams of dye in the tank with respect to the time t.

Solution

First write the function $A(t)$ as

$$A(t) = 20 + t - \frac{4000}{(20+t)^2}$$

Then the rate of change is

$$\frac{dA}{dt} = 1 + \frac{8000}{(20+t)^3}$$

(b) Determine the initial rate of change of the number of grams of dye in the tank. Explain why your value makes sense.

Solution

The initial rate of change is found by evaluating the rate of change above at $t = 0$. This gives

$$\left.\frac{dA}{dt}\right|_{t=0} = 1 + \frac{8000}{20^3} = 2\,\text{gm/min}$$

This makes sense since the solution being pumped in adds 3 grams of dye per minute (1 g/L $\times$ 3 L/min). The initial concentration of dye is 0.5 grams per litre and the solution is being pumped out at 2 L/min, so the solution being pumped out takes 1 gram per minute. This gives a net increase of 2 grams per minute.

(c) Determine the number grams of dye in the tank when it just starts to run over and the rate of change of the number of grams of dye in the tank at this time.

Solution

Since the dye mixture is pumped in at 3 L/min and the is pumped out at 2 L/min, the tank is filling at the rate of 1 L/min. Hence, the tank just starts to overflow after 30 minutes. At this time the amount of dye in the tank is given by

$$A(30) = 20 + 30 - \frac{4000}{50^2} = 50 - 1.6 = 48.4\,\text{g}$$

and the rate of change is

$$\left.\frac{dA}{dt}\right|_{t=30} = 1 + \frac{8000}{50^3} = 1 + 0.64 = 1.064\,\text{g/min}$$

Example 6 continued . . .

...Example 6 continued

(d) Show that $A\left(t\right)$ satisfies the **differential equation**

$$\frac{dA}{dt} = 3 - \frac{2A}{20+t}$$

and explain the meaning of the two terms on the right hand side.

Solution

From part (a)

$$\frac{dA}{dt} = 1 + \frac{8000}{(20+t)^3}$$

and using the alternate expression for $A(t)$ used in part (a)

$$\begin{aligned} 3 - \frac{2A}{20+t} &= 3 - \left(\frac{2}{20+t}\right)\left[20 + t - \frac{4000}{(20+t)^2}\right] = 3 - 2\left[\frac{20+t}{20+t} - \frac{4000}{(20+t)^3}\right] \\ &= 3 - 2 + \frac{8000}{(20+t)^3} = 1 + \frac{8000}{(20+t)^3} \end{aligned}$$

So that

$$\frac{dA}{dt} = 3 - \frac{2A}{20+t}$$

This means that the rate of change of the number of grams of dye in the tank is made up of two parts. One part from the dye mixture flowing in at 3 L/min with a concentration of 1 g/L, which gives an increase of 1 g/min in the dye in the tank. The other part of the rate of change of the number of grams of dye in the tank comes from the dye mixture being pumped out at the 2 L/min. The concentration of dye in the mixture is

$$\frac{\text{number of grams of dye in tank}}{\text{amount of water in tank}} = \frac{A}{20+t}$$

since the amount of water in the tank is increasing from an initial amount of 20 L at 1 L/min. This concentration is in g/L. Thus, the rate at which dye is pumped out of the tank is

$$\left(\frac{A}{20+t}\right)\left(\frac{\text{g}}{\text{L}}\right)\left[2\left(\frac{\text{L}}{\text{min}}\right)\right] = \frac{2A}{20+t}\frac{\text{g}}{\text{min}}$$

Example 7

A seventy-metre high tower casts a shadow on level ground. As the sun sets its angle of elevation decreases at 18 degrees per hour. How fast is the tower's shadow lengthening when the sun is 30 degrees above the horizon?

Solution
Let s be the length of the shadow and θ be the angle the sun is above the horizon. We are given

$$\frac{d\theta}{dt} = -18^\circ/\text{h} = -\frac{\pi}{10}\,\text{h}^{-1}$$

Note that rate of change is negative, since the sun is going down. We are asked to find

$$\frac{ds}{dt} \text{ when } \theta = 30^\circ$$

A relation between s and θ is given by simple trigonometry as

$$\frac{70}{s} = \tan\theta \Rightarrow s = \frac{70}{\tan\theta} = 70\cot\theta$$

Then taking the derivative of both sides with respect to t gives

$$\frac{ds}{dt} = -70\csc^2\theta\frac{d\theta}{dt} = -\frac{70}{\sin^2\theta}\frac{d\theta}{dt}$$

Note we can also get this result using the expression in terms of $\tan\theta$ as

$$\frac{ds}{dt} = -\frac{70\sec^2\theta}{\tan^2\theta} = -\frac{70}{\left(\dfrac{1}{\cos^2\theta}\right)}\left(\frac{\cos^2\theta}{\sin^2\theta}\right) = -\frac{70}{\sin^2\theta}$$

In any case, plugging in the given value for $\dfrac{d\theta}{dt}$ and the value of $\theta = 30^\circ$ for the particular instant gives

$$\frac{ds}{dt} = -\frac{70}{\left(\frac{1}{2}\right)^2}\left(\frac{\pi}{10}\right) = 28\pi\,\text{m/h} \approx 88\,\text{m/h}$$

Example 8

When a gas expands in such a way that it does not exchange heat with its surroundings, it is undergoing **adiabatic expansion**. For a certain gas undergoing adiabatic expansion the relation between the volume V and the pressure P is $PV^{1.3} = k$ where k is constant.

(a) Show that the rate of change of the volume V with respect to the pressure P is $-\dfrac{V}{1.3P}$.

Solution
Taking derivative of both sides of the equation relating P and V with respect to P, treating V as a function of P, gives

$$V^{1.3} + P\left(1.3V^{0.3}\frac{dV}{dP}\right) = 0 \Rightarrow 1.3PV^{0.3}\frac{dV}{dP} = -V^{1.3}$$
$$\Rightarrow \frac{dV}{dP} = -\frac{V^{1.3}}{1.3PV^{0.3}} = -\frac{V}{1.3P}$$

This is the rate of change of the volume V with respect to the pressure P.

Example 8 continued . . .

...Example 8 continued

(b) The **compressibility** of a gas is defined as $\beta = -\frac{1}{V}\frac{dV}{dP}$. Show that for the gas above the compressibility is

$$\beta = \frac{1}{1.3P}$$

Solution

The compressibility is

$$\beta = -\frac{1}{V}\left(-\frac{V}{1.3P}\right) = \frac{1}{1.3P}$$

(c) A gas is contained in a thermally isolated cylinder so that it does not exchange heat with its surroundings. The cylinder is fitted with a movable piston that can slide up or down to change the volume. Initially the gas has a volume of 12 L. The pressure in the cylinder is initially 300 kPa and is decreasing at 4 kPa per minute. How fast is the volume of the gas in the cylinder changing? Is the volume increasing or decreasing?

Solution

Here we are give $\frac{dP}{dt} = -4\,\text{kPa/min}$. We are asked to find $\frac{dV}{dt}$ when $V = 12\,\text{L}$ and $P = 300\,\text{kPa}$. From part (a)

$$\frac{dV}{dt} = \frac{dV}{dP}\frac{dP}{dt} = -\frac{V}{1.3P}\frac{dP}{dt}$$

Plugging in the given data yields

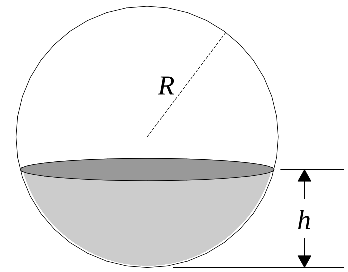
$$\frac{dV}{dt} = -\frac{12\,\text{L}}{1.3\,(300\,\text{kPa})}\,(-4\,\text{kPa/mi}) = \frac{48}{390}\,\text{L/min} = 0.123\,\text{L/min}$$

Note that no conversion of the volume units was required. The volume is increasing as the pressure is decreasing. As we should expect.

Example 9

A spherical segment is formed by cutting a sphere perpendicular to the axis. See the diagram. The volume of a spherical segment of depth h cut from a sphere of radius R is

$$V = \frac{\pi}{3} h^2 (3R - h)$$



A spherical tank has a 20 cm radius. It is initially filled with water to a depth of 25 cm. The water leaks out of the a small hole in the bottom of the tank at a rate of 50 mL/s.

(a) Find the rate at which the depth of the water in the tank is changing when the water in the tank is 25 cm deep.

Solution

We are given $\frac{dV}{dt} = 50\,\text{mL/s} = 50\,\text{cm}^3/\text{s}$. We are asked to find $\frac{dh}{dt}$ when $h = 25\,\text{cm}$. We are given the relation between V and h. Write it as

$$V = \frac{\pi}{3}\left(3Rh^2 - h^3\right)$$

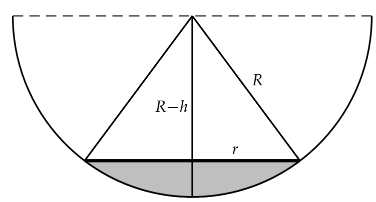
Then take the derivative with respect to t to give

$$\frac{dV}{dt} = \frac{\pi}{3}\left(6Rh - 3h^2\right)\frac{dh}{dt} = \pi h (2R - h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi h (2R - h)}\frac{dV}{dt}$$

Plugging in gives

$$\left.\frac{dh}{dt}\right|_{h=25\,\text{cm}} = \frac{50}{\pi 25 (40 - 25)} = \frac{2}{15\pi} = 0.042\,\text{cm/s}$$

(b) Use the diagram shown to find a relation between the depth h of the water in the tank and radius r of the surface of the water in the tank. Then use this relation and the rate of change found in part (a) above to find the rate at which the area of the surface of the water in the tank is changing when the water in the tank is 25 cm deep.



Solution

From the diagram we see that

$$r^2 + (R - h)^2 = R^2 \Rightarrow r^2 + R^2 - 2Rh + h^2 = R^2 \Rightarrow r^2 = 2Rh - h^2$$

Then the area of the surface of the water is

$$A = \pi r^2 = \pi\left(2Rh - h^2\right)$$

and the rate of change of the area is

$$\frac{dA}{dt} = \pi (2R - 2h)\frac{dh}{dt} = 2\pi (R - h)\frac{dh}{dt}$$

Substituting for $\frac{dh}{dt}$ from part (a) gives

$$\frac{dA}{dt} = 2\pi (R - h)\left[\frac{1}{\pi h (2R - h)}\frac{dV}{dt}\right] = \frac{2(R - h)}{h(2R - h)}\frac{dV}{dt}$$

Plugging in gives

$$\left.\frac{dA}{dt}\right|_{h=25} = \frac{100(20 - 25)}{25(40 - 25)} = -\frac{4}{3} = -1.33\,\text{cm}^2/\text{s}$$