Example 1

Let $y = \dfrac{\left(x^3-8\right)^3}{\left(x^3+8\right)^2}$.

(a) Use the properties of logarithms to simplify $\ln y$.

(b) Use implicit differentiation on the simplified expression for $\ln y$ from part (a) to find $\dfrac{dy}{dx}$. Express the final result in terms of x alone.

Example 2

Use logarithmic differentiation to find the derivative of each of following functions:

(a) $y = xe^{\sin x}$

(b) $y = x^{\cos x}$

Example 3

Find the point(s) on the curve $y = x^x$ where the tangent line is horizontal.

Example 4

When fluid flows through a pipe the radius r of the pipe can be determined by measuring the flow rate R. This could be useful in determining the radii of capillary blood vessels. Under certain conditions the radius is given in terms of the flow rate by the formula

$$r = \left(\frac{R}{25}\right)^{1/4}$$

where R is measured in cm^3/s and r is measured in centimetres.

(a) Calculate the average rate of change of r with respect to the flow rate over the intervals $300 \le R \le 400$ and $400 \le R \le 500$.

(b) Find the instantaneous rate of change of the radius r of the pipe with respect to flow rate in when the flow rate is $R = 400\,\text{cm}^3/\text{s}$. Give the units of the rate of change and explain in practical terms what it means.

Example 5

The mass of a 5-metre tapered beam varies in such a way that the total mass in kilograms between the thick end and a point x metres from the end is

$$m(x) = 70 - \frac{140}{x+2} \quad \text{for} \quad 0 \le x \le 5$$

This is the **mass function** of the beam.

(a) What is the total mass of the beam?

(b) The rate of change of the total mass function with respect to the distance for the thick end gives the **linear mass density**. Find the linear mass density of the beam as a function of x.

(c) Find the linear mass density at

i) the thick end of the beam

ii) the thin end of the beam

iii) the middle of the beam

(d) What is the average density of the beam? Is there any point on the beam where the linear mass density equals this average density?

Example 6

A fifty litre tank contains twenty litres of water that has 10 grams of a certain dye dissolved in it. Water containing 1 g/L of the dye is pumped into the tank at 3 L/min and the well mixed dye solution is pumped out at 2 L/min. The total number of grams of dye in the tank after t minutes is given by

$$A(t) = \frac{(20+t)^3 - 4000}{(20+t)^2}$$

(a) Find the rate of change of the number of grams of dye in the tank with respect to the time t.

(b) Determine the initial rate of change of the number of grams of dye in the tank. Explain why your value makes sense.

(c) Determine the number grams of dye in the tank when it just starts to run over and the rate of change of the number of grams of dye in the tank at this time.

(d) Show that $A(t)$ satisfies the **differential equation**

$$\frac{dA}{dt} = 3 - \frac{2A}{20+t}$$

and explain the meaning of the two terms on the right hand side.

Example 7

A seventy-metre high tower casts a shadow on level ground. As the sun sets its angle of elevation decreases at 18 degrees per hour. How fast is the tower's shadow lengthening when the sun is 30 degrees above the horizon?

Example 8

When a gas expands in such a way that it does not exchange heat with its surroundings, it is undergoing **adiabatic expansion**. For a certain gas undergoing adiabatic expansion the relation between the volume V and the pressure P is $PV^{1.3} = k$ where k is constant.

(a) Show that the rate of change of the volume V with respect to the pressure P is $-\dfrac{V}{1.3P}$.

(b) The **compressibility** of a gas is defined as $\beta = -\dfrac{1}{V}\dfrac{dV}{dP}$. Show that for the gas above the compressibility is

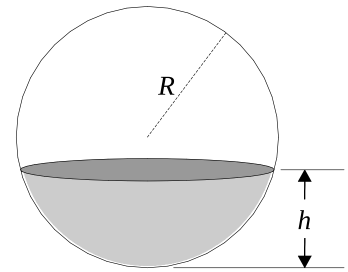
$$\beta = \frac{1}{1.3P}$$

(c) A gas is contained in a thermally isolated cylinder so that it does not exchange heat with its surroundings. The cylinder is fitted with a movable piston that can slide up or down to change the volume. Initially the gas has a volume of 12 L. The pressure in the cylinder is initially 300 kPa and is decreasing at 4 kPa per minute. How fast is the volume of the gas in the cylinder changing? Is the volume increasing or decreasing?

Example 9

A spherical segment is formed by cutting a sphere perpendicular to the axis. See the diagram. The volume of a spherical segment of depth h cut from a sphere of radius R is

$$V = \frac{\pi}{3}h^2\,(3R - h)$$



A spherical tank has a 20 cm radius. It is initially filled with water to a depth of 25 cm. The water leaks out of the a small hole in the bottom of the tank at a rate of 50 mL/s.

(a) Find the rate at which the depth of the water in the tank is changing when the water in the tank is 25 cm deep.

(b) Use the diagram shown to find a relation between the depth h of the water in the tank and radius r of the surface of the water in the tank. Then use this relation and the rate of change found in part (a) above to find the rate at which the area of the surface of the water in the tank is changing when the water in the tank is 25 cm deep.