Okanagan College

Math 112 (071) Fall 2009

Term Test Two – Problems & Solutions

Instructor: Clint Lee

Wednesday, October 29

Student Name: ____________________ Total Marks: ________ / 40

Instructions. Do all parts of all 10 questions. Show all work and give explanations where required. You may receive part marks for a question if your work is correct even if the final answer is incorrect. However, if your answer is incorrect and no work or explanation is given, you will receive no marks. The number of points for each question is given in the left margin, total 40.

In problems 1 through 5 give a brief answer. You will be marked only on your answer, not on your work.

[2] 1. Evaluate the limit or explain why it does not exist: $\lim_{x\to 0} \dfrac{x^2 - x - 12}{x^2 - 7x + 12}$

Solution

Using direct substitution gives

$$\lim_{x\to 0} \frac{x^2 - x - 12}{x^2 - 7x + 12} = \frac{-12}{12} = -1$$

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[2] 2. Evaluate the limit or explain why it does not exist: $\lim_{x\to\infty} \dfrac{3x^2 + 4}{x^2 - 1}$

Solution

The limit at infinity of a rational function with the highest power in the numerator equal to the highest power in the denominator equals the ratio of the coefficients of the highest powers. Or

$$\lim_{x\to\infty} \frac{3x^2 + 4}{x^2 - 1} = \lim_{x\to\infty} \frac{3 + \dfrac{4}{x^2}}{1 - \dfrac{1}{x^2}} = 3$$

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[2] 3. Find $f'(t)$ for $f(t) = t^2 + 2e^t$. Do not simplify.

Solution

Using the Power and Exponential Function Rules

$$f'(t) = 2t + 2e^t$$

[2] 4. Find $k'(x)$ for $k(x) = \dfrac{x}{x+1}$. Do not simplify.

Solution

Using the Quotient Rule

$$k'(x) = \frac{(1)(x+1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2}$$

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[2] 5. Let $y = f(x)$ where $\left.\dfrac{dy}{dx}\right|_2 = 3$ and $f(2) = 4$. Find an equation of the tangent line to the curve $y = f(x)$ at the point on the curve where $x = 2$.

Solution

The slope of the tangent line is $m = \left.\dfrac{dy}{dx}\right|_2 = 3$ and the point of tangency is $(2, f(2)) = (2, 4)$. So the equation of the tangent line is

$$y - 4 = 3(x-2) = 3x - 6 \Rightarrow y = 3x - 2$$

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[3] 6. A version of the Intermediate Value Theorem is given below with some parts left blank. Fill in each blank with the correct quantity or statement from the selections (A) – (I) given below.

Let f be a function that is <u>(B), continuous</u> on the interval $[a, b]$. Suppose that $f(a) > 0$ and $f(b)$ <u>(G), < 0</u>. There is a number c in the interval (a, b) for which <u>(D), $f(c) = 0$</u>.

(A) differentiable	(B) continuous	(C) defined
(D) $f(c) = 0$	(E) $f'(c) = 0$	(F) $f\left(\dfrac{a+b}{2}\right) = c$
(G) < 0	(H) > 0	(I) $= 0$

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7. Evaluate each limit, if it exists. If the limit is infinite, give its value as either ∞ or $-\infty$. If the limit does not exist, explain why.

[3] (a) $\displaystyle\lim_{x\to 4} \frac{x^2 - x - 12}{x^2 - 7x + 12}$

Solution

Direct substitution gives $\frac{0}{0}$. Factor and cancel to give

$$\lim_{x\to 4} \frac{x^2 - x - 12}{x^2 - 7x + 12} = \lim_{x\to 4} \frac{(x-4)(x+3)}{(x-4)(x-3)} = \lim_{x\to 4} \frac{x+3}{x-3} = 7$$

Problem 7 continued ...

... Problem 7 continued

[3] (b) $\lim_{x\to 2} \dfrac{x^2+x-6}{\sqrt{x+2}-2}$

Solution

Direct substitution gives $\frac{0}{0}$. Let $u = \sqrt{x+2}$, so that $x = u^2 - 2$ and $x \to 2 \Rightarrow u \to 2$. Then

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2+x-6}{\sqrt{x+2}-2} &= \lim_{u\to 2} \frac{x^2+x-6}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{\sqrt{x+2}-2} \\
&= \lim_{u\to 2} \frac{\left(u^2-4\right)\left(u^2+1\right)}{u-2} = \lim_{u\to 2} \frac{(u-2)(u+2)\left(u^2+1\right)}{u-2} \\
&= \lim_{u\to 2} (u+2)\left(u^2+1\right) = (4)(5) = 20
\end{aligned}$$

[3] (c) $\lim_{x\to 3^-} \dfrac{1-x}{\sqrt{10-x^2}-1}$

Solution

Direct substitution gives $\frac{-2}{0}$. Hence, this is an infinite limit. For $x < 3$, but close to $x = 3$

$$1 - x < 0 \text{ and } 10 - x^2 > 1 \Rightarrow \sqrt{10-x^2} - 1 > 0$$

Hence,

$$\lim_{x\to 3^-} \frac{1-x}{\sqrt{10-x^2}-1} = \frac{-}{+} = -\infty$$

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8. Find the indicated derivative, or derivatives. Simplify only if instructed to do so.

[3] (a) Find $\left.\dfrac{ds}{dt}\right|_{t=64}$ for $s = \dfrac{\sqrt{t}}{2} + \dfrac{3}{\sqrt[3]{t}}$.

Solution

First rewrite s as

$$s = \frac{\sqrt{t}}{2} + \frac{3}{\sqrt[3]{t}} = \tfrac{1}{2}t^{1/2} + 3t^{-1/3}$$

Then the derivative is

$$\frac{ds}{dt} = \tfrac{1}{4}t^{-1/2} - t^{-4/3} = \frac{1}{4\sqrt{t}} - \frac{1}{t\sqrt[3]{t}}$$

and evaluating the derivative at $t = 64$ gives

$$\left.\frac{ds}{dt}\right|_{t=64} = \frac{1}{4(8)} - \frac{1}{64(4)} = \frac{1}{32} - \frac{1}{256} = \frac{7}{256}$$

Problem 8 continued ...

. . . Problem 8 continued

[5] (b) Find $q'(x)$ and $q''(x)$ for $q(x) = \left(x^2 + 2x\right) e^{-2x}$. Simplify both derivatives.

Solution

Using the Product, Power, and Exponential Function Rules gives

$$\begin{aligned} q'(x) &= (2x+2)\, e^{-2x} + \left(x^2 + 2x\right)\left(-2e^{-2x}\right) = \left(2x + 2 - 2x^2 - 4x\right) e^{-2x} \\ &= \left(2 - 2x - 2x^2\right) e^{-2x} = 2\left(1 - x - x^2\right) e^{-2x} \end{aligned}$$

Taking the derivative again gives

$$\begin{aligned} q''(x) &= 2\left[(-1-2x)\, e^{-2x} + \left(1 - x - x^2\right)\left(-2e^{-2x}\right)\right] = 2\left(-1 - 2x - 2 + 2x + 2x^2\right) e^{-2x} \\ &= 2\left(2x^2 - 3\right) e^{-2x} \end{aligned}$$

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[2] 9. (a) State the limit definition of the derivative of the function f.

Solution

The limit definition is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

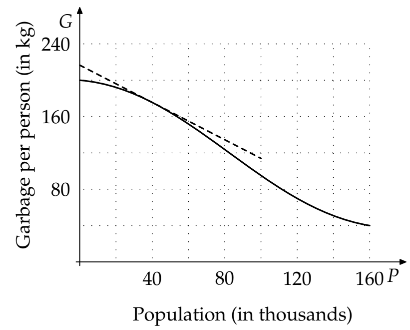
[4] (b) For the function in Problem 4, $k(x) = \dfrac{x}{x+1}$, use the limit definition of the derivative to find $k'(x)$. Completely simplify your answer and make sure that it agrees with your answer in Problem 4.

Solution

Applying the definition above gives

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h \to 0} \frac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} \\ &= \lim_{h \to 0} \frac{1}{h}\left(\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}\right) \\ &= \lim_{h \to 0} \frac{x^2 + x + xh + h - x^2 - xh - x}{h(x+h+1)(x+1)} \\ &= \lim_{h \to 0} \frac{h}{h(x+h+1)(x+1)} = \lim_{h \to 0} \frac{1}{(x+h+1)(x+1)} \\ &= \frac{1}{(x+1)^2} \end{aligned}$$

[4] 10. Studies show that the amount of garbage produced per household in small rural communities decreases as the population increases. The results of one study produced the results shown in the graph at the right. If $G(P)$ is the number of kilograms of garbage produced per person in one month when the population is P thousand people, draw an appropriate tangent line on the graph shown to estimate the value of $G'(40)$. Explain in practical terms what the value of this derivative represents and give its units.



Solution

The tangent line is shown. Reading off the coordinates of the endpoints of the tangent line drawn as $(0, 215)$ and $(100, 118)$ gives the slope of the tangent line as

$$m = \frac{118 - 215}{100 - 0} = -0.97$$

The units of this derivative are kilograms per person per month per thousand people. It means that when the population increases by 1000 from 40,000 the amount of garbage produced decreases by 0.97 kilograms per person per month.