Okanagan College

Math 112 (071) Winter 2009

Term Test Two – Problems & Solutions

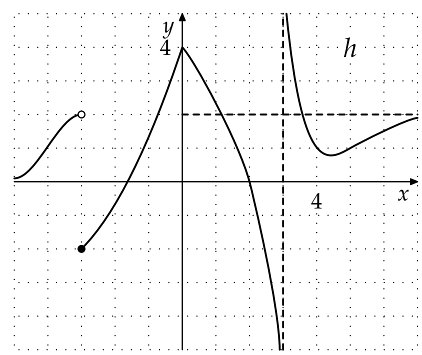
Instructor: Clint Lee

Thursday, March 12

Student Name: ______________________ Total Marks: ________ / 40

Instructions. Do all parts of all 6 questions. Show all work and give explanations where required. You may receive part marks for a question if your work is correct even if the final answer is incorrect. However, if your answer is incorrect and no work or explanation is given, you will receive no marks. The number of points for each question is given in the left margin, total 40.

1. The graph of a function h is shown. In parts (a) to (e) give the value of each quantity, or explain why the quantity does not exist.



[1] (a) $\lim_{x \to 0} h(x) = 4$

Solution

The function h is continuous at $x = 0$, just read off value of h from graph.

[2] (b) $\lim_{x \to 4^+} h(x) = \infty$

Solution

There is a vertical asymptote at $x = 4$ and the graph goes up the vertical asymptote on the right. Hence, the limit is $+\infty$.

[1] (c) $\lim_{x \to -\infty} h(x) = 0$

Solution

There is a horizontal asymptote on the left along the line $y = 0$, the x-axis.

[1] (d) $h(-3) = -2$

Solution

From the closed circle

[2] (e) Identify all of the vertical and horizontal asymptotes of the graph of h. Draw each asymptote on the given graph of h.

Solution

There is a vertical asymptote at $x = 4$. There is a horizontal asymptote on the left at $y = 0$ and on the right at $y = 2$.

2. Evaluate each limit, or explain why the limit does not exist. If the limit is infinite, express the limit as $\pm\infty$ or explain why it does not exist.

[3] (a) $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x^2+4}$

Solution

Direct substitution gives

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2+4} = \frac{4+2-6}{4+4} = \frac{0}{8} = 0$$

[3] (b) $\displaystyle\lim_{x\to -1} \frac{x^2+4x+3}{2x^2+x-1}$

Solution

Direct substitution gives $\frac{0}{0}$. Here we can factor and cancel to give

$$\begin{aligned}\lim_{x\to -1} \frac{x^2+4x+3}{2x^2+x-1} &= \lim_{x\to -1} \frac{(x+1)(x+3)}{(2x-1)(x+1)}\\ &= \lim_{x\to -1} \frac{x+3}{2x-1} = -\frac{2}{3}\end{aligned}$$

[3] (c) $\displaystyle\lim_{x\to -3^+} \frac{x+1}{x^2-9}$

Solution

Direct substitution gives $\frac{1}{0}$, so this is an infinite limit. For $-3 < x < 3$ we have $x+1 < 0$ and $x^2-9 < 0$. So that

$$\lim_{x\to -3^+} \frac{x+1}{x^2-9} = +\infty$$

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3. Let

$$p(x) = \frac{2x}{x-2}$$

[4] (a) Use the limit definition of the derivative

$$p'(x) = \lim_{h\to 0} \frac{p(x+h)-p(x)}{h}$$

to find a formula for $p'(x)$.

Solution

Using the definition gives

$$\begin{aligned}p'(x) &= 2\lim_{h\to 0} \frac{\dfrac{x+h}{x+h-2} - \dfrac{x}{x-2}}{h} = 2\lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-2)-x(x+h-2)}{(x+h-2)(x-2)}\right)\\ &= 2\lim_{h\to 0} \frac{1}{h}\left(\frac{x^2-2x+hx-2h-x^2-xh+2x}{(x+h-2)(x-2)}\right) = 2\lim_{h\to 0} \frac{1}{h}\left(\frac{-2h}{(x+h-2)(x-2)}\right)\\ &= -4\lim_{h\to 0} \frac{1}{(x+h-2)(x-2)} = -\frac{4}{(x-2)^2}\end{aligned}$$

Problem 3 continued ...

. . . Problem 3 continued

[2] (b) Use differentiation rules to verify your answer in part (a) above for $p'(x)$.

Solution

Using the Quotient Rule gives

$$p'(x) = 2\frac{(x-2)-x}{(x-2)^2} = 2\frac{-2}{(x-2)^2} = -\frac{4}{(x-2)^2}$$

[2] (c) Use your answer in parts (a) or (b) to find the equation of the line tangent to the graph of p where $x = 4$.

Solution

For $x = 4$ we have $y = 4$ and the slope of the tangent line is

$$m_T = p'(4) = -\frac{4}{2^2} = -1$$

So the equation of the tangent line is

$$y - 4 = (-1)(x-4) = -x + 4 \Rightarrow y = -x + 8$$

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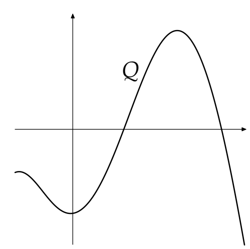
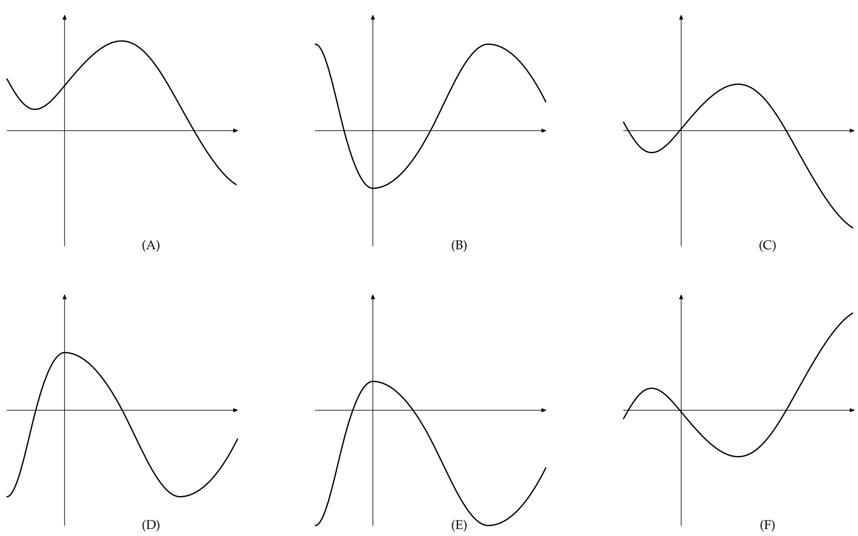
[3] 4. The graph of a function Q is shown at the right. The graphs of six other functions are shown below. Select the one graph that is the graph of the derivative Q'. Explain your choice.





Solution

Graph (C) is the graph of the Q'. To see this note that Q has a low point at $x = 0$, corresponding to Q' being zero at $x = 0$. Further, Q has negative slope for $x < 0$ and positive slope for $x > 0$, until the high point. This corresponds to negative values for Q for $x < 0$ and positive for $x > 0$. The only graph having these properties is graph (C).

5. Find the indicated derivative(s) for each function. Simplify only if required.

[3] (a) Find $\frac{dy}{dx}$ for $y = 6\sqrt[3]{x^2} + 2e^{-3x} - \frac{3}{x^5}$.

Solution

Write the function as

$$y = 6x^{2/3} + 2e^{-3x} - 3x^{-5}$$

Then

$$\frac{dy}{dx} = 6\left(\tfrac{2}{3}\right)x^{-1/3} + 2(-3)e^{-3x} - 3(-5)x^{-6} = 4x^{-1/3} - 6e^{-3x} + 15x^{-6} = \frac{4}{\sqrt[3]{x}} - 6x^{-3x} + \frac{15}{x^6}$$

[5] (b) Find and simplify $g'(x)$ and $g''(x)$ for $f(x) = \left(x^2 - 2x + 3\right)e^x$.

Solution

Using the Product Rule gives

$$g'(x) = (2x-2)\,e^x + \left(x^2 - 2x + 3\right)e^x = \left(x^2+1\right)e^x$$

Using the Product Rule again gives

$$g''(x) = 2xe^x + \left(x^2+1\right)e^x = \left(x^2 + 2x + 1\right)e^x = (x+1)^2\,e^x$$

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6. A chemistry student is doing a summer study of the mercury concentrations in a small alpine lake. Runoff from the surrounding snow pack deposits mercury in the lake during the height of the summer, but as the runoff decreases some of the mercury is flushed out of the lake as a result of the stream that flows out of the lake. The student takes readings every 10 days during the 90 day study. The data in the table below gives the average mercury concentration C, in mg/m^3 (milligrams per cubic metre), from samples taken at several locations and depths in the lake.

t, days	0	10	20	30	40	50	60	70	80	90
C, mg/m^3	100	108	125	132	126	105	98	90	85	83

[2] (a) Give the average rate of change of the mercury concentration in the lake during the first 30 days of the study, that is, from $t = 0$ to $t = 30$. Give the units of this rate of change.

Solution

The average rate of change is

$$\text{average ROC} = \frac{132-100}{30-0} = \frac{32}{30} = 1.07\,\frac{\text{mg/m}^3}{\text{day}}$$

[3] (b) Estimate the instantaneous rate of change of the mercury concentration in the lake after 50 days, that is, at $t = 50$. Explain what this rate of change tells you.

Solution

Estimate the instantaneous rate of change at $t = 50$ by straddling this time, that means taking the average rate of change over the interval from $t = 40$ to $t = 60$. This gives

$$\left.\frac{dC}{dt}\right|_{t=50} \approx \frac{98-126}{60-40} = -\frac{7}{5} = -1.4\,\frac{\text{mg/m}^3}{\text{day}}$$

This means that from day 50 to day 51 the mercury concentration has decreased by 1.4 mg/m^3.