Okanagan College

Math 122 (71) Winter 2009

Term Test One – Problems & Solutions

Instructor: Clint Lee

Wednesday, February 4

Student Name: ____________________ Total Marks: ________ / 40

Instructions. Do all parts of all 6 questions. Show all work and give explanations where required. You may receive part marks for a question if your work is correct even if the final answer is incorrect. However, if your answer is incorrect and no work or explanation is given, you will receive no marks. The number of points for each question is given in the left margin, total 40. A Formula Sheet is attached. You may use any formula on this sheet on any problem on this test.

1. Evaluate each integral. For any definite integral, give the exact numerical value.

[3] (a) $\displaystyle\int \left(\frac{2}{x^2} + \frac{3}{2x+1} + e^{x/2}\right) dx$

Solution
Rewriting the integral slightly

$$\begin{aligned}
\int \left(\frac{2}{x^2} + \frac{3}{2x+1} + e^{x/2}\right) dx &= \int \left(2x^{-2} + \frac{3}{2x+1} + e^{x/2}\right) dx \\
&= -2x^{-1} + \tfrac{3}{2}\ln|2x+1| + \frac{1}{\left(\frac{1}{2}\right)}e^{x/2} + C \\
&= -\frac{2}{x} + \tfrac{3}{2}\ln|2x+1| + 2e^{x/2} + C
\end{aligned}$$

[3] (b) $\displaystyle\int \frac{\cos(2+\ln x)}{x}\, dx$

Solution
Making the substitution

$$u = 2 + \ln x \Rightarrow du = \frac{1}{x}\, dx$$

gives

$$\begin{aligned}
\int \frac{\cos(2+\ln x)}{x}\, dx &= \int \cos(2+\ln x)\left(\frac{1}{x}\, dx\right) \\
&= \int \cos u\, du = \sin u + C = \sin(2+\ln x) + C
\end{aligned}$$

. . . Problem 1 continued

[3] (c) $\displaystyle\int_0^2 \frac{t^3}{\sqrt{4+2t^4}}\,dt$

Solution

Making the substitution

$$u = 4 + 2t^4 \Rightarrow du = 8t^3\,dt \Rightarrow t^3\,dt = \tfrac{1}{8}\,du$$

and changing the limits of integration

$$x = 2 \Rightarrow u = 36$$
$$x = 0 \Rightarrow u = 4$$

gives

$$\int_0^2 \frac{t^3}{\sqrt{4+2t^4}}\,dt = \int_0^2 \frac{1}{\sqrt{4+2t^4}}\left(t^3\,dt\right)$$
$$= \tfrac{1}{8}\int_4^{36} u^{-1/2}\,du = \tfrac{1}{8}\left(2u^{1/2}\right)\Big|_4^{36} = \tfrac{1}{4}\,(6-2) = 1$$

---

2. Consider the definite integral

$$\int_2^5 \left(x^3 - 4x\right)\,dx$$

[2] (a) Set up the right endpoint Riemann sum for the definite integral above using n subintervals, R_n. Do not simplify

Solution

With n subintervals on the interval $[2,5]$,

$$\Delta x = \frac{5-2}{n} = \frac{3}{n} \Rightarrow x_i = 2 + \frac{3i}{n}$$

Hence, the Riemann sum is

$$R_n = \sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)^3 - 4\left(2+\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right)$$

[3] (b) Using Maple the sum in part (a) can be evaluated and simplified to

$$R_n = \frac{63}{4}\frac{(n+1)(7n+3)}{n^2}$$

Use an appropriate limit to determine the exact value of the definite integral above.

Solution

Expanding and dividing through by n^2 gives

$$R_n = \frac{63}{4}\frac{7n^2+10n+21}{n^2} = \frac{63}{4}\left(7 + \frac{10}{n} + \frac{21}{n^2}\right)$$

So that

$$\int_2^5 \left(x^3-4x\right)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{63}{4}\left(7+\frac{10}{n}+\frac{21}{n^2}\right)$$
$$= \frac{63}{4}\lim_{n\to\infty}\left(7+\frac{10}{n}+\frac{21}{n^2}\right) = \frac{63}{4}\,(7) = \frac{441}{7}$$

Problem 2 continued . . .

...Problem 2 continued

[2] (c) Evaluate the definite integral above using the Fundamental Theorem of Calculus to verify your answer in part (b).

Solution

$$\int_2^5 \left(x^3 - 4x\right)\, dx = \left(\tfrac{1}{4}x^4 - 2x^2\right)\Big|_2^5 = \frac{625}{4} - 50 - 4 + 8 = \frac{441}{4}$$

---

[4] 3. Consider the definite integral

$$\int_4^7 \frac{x+1}{\sqrt{8-x}}\, dx$$

Evaluate this integral using the substitution $u = \sqrt{8-x}$.

Solution

Making the given substitution

$$u = \sqrt{8-x} \Rightarrow x = 8 - u^2 \Rightarrow dx = -2u\, du$$

Changing the limits of integration

$$x = 7 \Rightarrow u = 1$$
$$x = 4 \Rightarrow u = 2$$

Then the integral is

$$\begin{aligned}\int_4^7 \frac{x+1}{\sqrt{8-x}}\, dx &= -\int_2^1 \frac{\left(8-u^2\right)+1}{u}\ (-2u\, du) \\ &= 2\int_1^2 \left(9 - u^2\right)\, du \\ &= 2\left(9u - \tfrac{1}{3}u^3\right)\Big|_1^2 = 2\left(18 - \tfrac{1}{3}(8) - 9 + \tfrac{1}{3}\right) = \tfrac{40}{3}\end{aligned}$$

---

4. Wine is being drained from a fermentation vessel in a winery. There is a meter at the outlet from the vessel that measures the rate of flow out of the vessel. The flow rate decreases with time and the rate of decrease is also decreasing. The flow rate $r(t)$ is measured every 2 minutes over the 16 minutes it takes to drain the tank to the desired level. The measurements, in litres per minute, are given in the table below.

t, min	0	2	4	6	8	10	12	14	16
$r(t)$, L/min	52	44	37	31	26	22	19	17	16

[2] (a) Set up a definite integral for the total amount of wine that is drained out of the fermentation vessel during the 16 minute period.

Solution

If $V(t)$ is the total volume of wine drained from the vessel, then, by the Net Change Theorem, the definite integral is

$$V(16) - V(0) = \int_0^{16} r(t)\, dt$$

Problem 4 continued ...

...Problem 4 continued

[3] (b) Use the Midpoint Rule with n as large as possible to approximate the value of the definite integral in part (a). Can you determine if this approximation an overestimate or an underestimate? Explain?

Solution

Using the Midpoint Rule we must take $n = 4$, so that $\Delta x = 4$ and

$$M_4 = (4)\,[f(2) + f(6) + f(10) + f(14)] = 4\,(44 + 31 + 22 + 17) = 456\,\mathrm{L}$$

Since the graph of $r(t)$ is concave up, the rate of decrease is decreasing, this gives an underestimate.

[3] (c) Use the Simpson's Rule with n as large as possible to approximate the value of the definite integral in part (a). Can you determine if this approximation an overestimate or an underestimate? Explain?

Solution

Using $n = 8$, so that $\Delta x = 2$, with Simpson's Rule gives

$$\begin{aligned} S_8 &= \tfrac{2}{3}\,[f(0) + 4f(2) + 2f(4) + 4f(6) + 2f(8) + 4f(10) + 2f(12) + 4f(14) + f(16)] \\ &= \tfrac{2}{3}\,[52 + 4(44) + 2(37) + 4(31) + 2(26) + 4(22) + 2(19) + 4(17) + 16] \\ &= \tfrac{2}{3}\,(688) = 458.7\,\mathrm{L} \end{aligned}$$

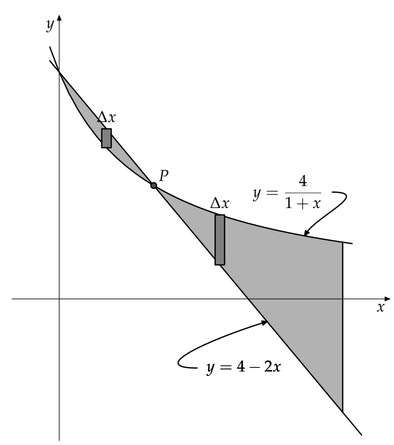
Alternatively

$$S_8 = \tfrac{1}{3}\,(2M_4 + T_4) = \tfrac{1}{3}\,[2(456) + 464]\,458.7\,\mathrm{L}$$

It is impossible to tell whether Simpson's Rule gives an over or underestimate from the information given.

---

5. The region bounded by the curves $y = \dfrac{4}{1+x}$ and $y = 4 - 2x$ over the interval from $x = 0$ to $x = 3$ is shaded in the diagram. It is required to find the area of this region.



[2] (a) Draw one or more typical approximating rectangles on the diagram and give an expression for the area of each.

Solution

The graphs cross over at the point P where $x = 1$ and $y = 2$. Hence, there are two separate regions so that two typical approximating rectangles are required. The areas of the rectangles are:

$$0 \le x < 1 : \ \Delta A = \left[4 - 2x - \frac{4}{1+x}\right] \Delta x$$

$$1 \le x < 3 : \ \Delta A = \left[\frac{4}{1+x} - (4 - 2x)\right] \Delta x = \left[\frac{4}{1+x} - 4 +\right] \Delta x$$

[3] (b) Set up one or more definite integrals that together give the required area. Do not evaluate the integral(s).

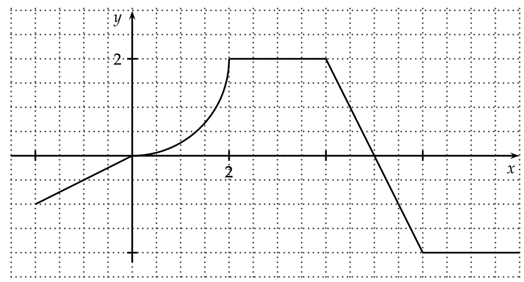
Solution

The integrals are

$$\int_0^1 \left[4 - 2x - \frac{4}{1+x}\right] dx + \int_1^3 \left[\frac{4}{1+x} - 4 +\right] dx$$

6. The graph of a function $f(t)$ is shown. The curved section of the graph is a circular arc. Define the function $h(x)$ as

$$h(x) = \int_{-2}^{x} f(t)\, dt$$



[2] (a) Find the values of $h(0)$, $h(2)$, $h(4)$, and $h(6)$.

Solution

Evaluating each definite integral as an area:

$$h(0) = \int_{-2}^{0} f(t)\, dt = -\tfrac{1}{2}(2)(1) = -1$$
$$h(2) = \int_{-2}^{2} f(t)\, dt = h(0) + \int_{0}^{2} f(t)\, dt = -1 + (2)(2) - \tfrac{1}{4}\pi\left(2^2\right) = 3 - \pi$$
$$h(4) = \int_{-2}^{4} f(t)\, dt = h(2) + \int_{2}^{4} f(t)\, dt = 3 - \pi + 4 = 7 - \pi$$
$$h(6) = \int_{-2}^{6} f(t)\, dt = h(4) + \int_{4}^{6} f(t)\, dt = 7 - \pi + 0 = 7 - \pi$$

[3] (b) Determine the intervals where the function h is increasing and decreasing. Identify any critical numbers of the function h and classify each as a local maximum, local minimum or neither. Give the absolute maximum and minimum values of the function h on the interval $[-2, 8]$.

Solution

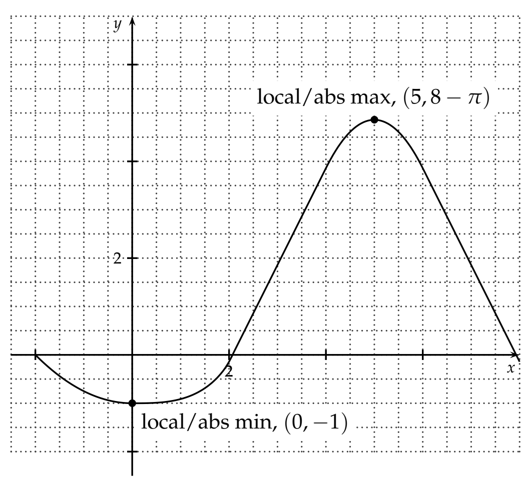
By the Fundamental Theorem of Calculus, Part I, $h'(x) = f(x)$. Hence, h is increasing where $f(x) > 0$, and decreasing where $f(x) < 0$. Therefore,

- h is increasing on the interval $(0, 5)$, and
- decreasing on $(-2, 0) \cup (5, 8)$.

There are critical numbers,

- At $x = 0$, a local minimum. Also the absolute minimum on $[-2, 8]$ with a value of $h(0) = -1$, since $h(0) < h(8) = 3 - \pi$.
- At $x = 5$, a local maximum. Also the absolute minimum on $[-2, 8]$ with a value of $h(5) = 8 - \pi$.

[2] (c) Sketch the graph of the function h. Label the local and absolute extremes identified in part (b) above.



Solution

The graph is shown.