Okanagan College

Math 122 (071) Winter 2010

Term Test One – Problems & Solutions

Instructor: Clint Lee

Wednesday, February 3

Student Name: ______________________________ Total Marks: ________ / 40

Instructions. Do all parts of all 11 questions. Show all work and give explanations where required. You may receive part marks for a question if your work is correct even if the final answer is incorrect. However, if your answer is incorrect and no work or explanation is given, you will receive no marks. The number of points for each question is given in the left margin, total 40.

In problems 1 through 6 give a brief answer. You will be marked only on your answer, not on your work.

[2] 1. Find the most general antiderivative of $f(x) = e^{2x+1} + \cos(2x+1)$.

Solution

The most general antiderivative is

$$F(x) = \tfrac{1}{2}e^{2x+1} + \tfrac{1}{2}\sin(2x+1) + C$$

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[2] 2. Find the antiderivative of the function $g(x) = \sqrt{\tfrac{1}{2}x + 3}$ whose graph goes through the point $(2, 4)$.

Solution

The most general antiderivative is

$$G(x) = \tfrac{2}{3}(2)\left(\tfrac{1}{2}x + 3\right)^{3/2} + C = \tfrac{4}{3}\left(\tfrac{1}{2}x + 3\right)^{3/2} + C$$

Using the fact that $G(2) = 4$ gives

$$G(2) = \tfrac{4}{3}(8) + C = \tfrac{32}{3} + C = 4 \Rightarrow C = 4 - \tfrac{32}{3} = -\tfrac{20}{3}$$

Hence,

$$G(x) = \tfrac{4}{3}\left(\tfrac{1}{2}x + 3\right)^{3/2} - \tfrac{20}{3}$$

[2] 3. Set up the n subinterval right-endpoint Riemann sum for the definite integral $\int_0^2 x^3\,\mathrm{d}x$.

Solution

Here $\Delta x = \frac{2}{n}$ and the right endpoint of the i^{th} subinterval is $x_i = \frac{2i}{n}$. Hence, the right endpoint Riemann sum is

$$R_n = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \frac{16}{n^4} \sum_{i=1}^{n} i^3$$

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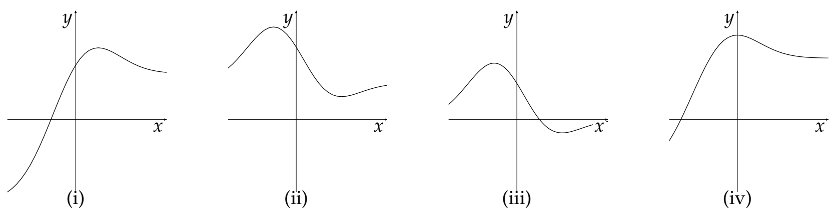
[2] 4. The graphs of four functions are shown. One is the graph of a function f and at least one other is the graph of an antiderivative of the function f. Identify the graph of the function f and the graph or graphs of an antiderivative. Explain your choices.



Solution

The function is **graph (iii)** and the only possible antiderivative is **graph (i)**. Let F be an antiderivative of f. Then $F'(x) = f(x)$, so that where $F(x)$ is increasing $f(x)$ is positive, where $F(x)$ has its maximum

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[2] 5. A function f is decreasing and concave up on the interval $[-2, 3]$. If you use R_{20}, L_{20}, T_{20}, M_{20}, and S_{20} to estimate the value of the definite integral $\int_{-2}^{3} f(x)\,\mathrm{d}x$, which give an overestimate and which an underestimate. Explain your answer.

Solution

Since f is decreasing on the interval, R_{20} is an underestimate and L_{20} is an overestimate. Since the graph of f is concave up on the interval M_{20} is an underestimate and T_{20} is an overestimate. It is impossible to tell from the given information whether S_{20} gives an over or underestimate since the error in Simpson's Rule depends on the fourth derivative of the function f.

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[2] 6. Fill in the two blanks in the statement of the Fundamental Theorem of Calculus, Part 2 given below:

Suppose that the function f is continuous on the interval $[a, b]$ and that F is an antiderivative of f, so that

$$F'(x) = \underline{\hspace{3em}}$$

Then

$$\int_a^b f(x)\,\mathrm{d}x = \underline{\hspace{6em}}$$

7. Water flows into my rain barrel during a downpour. Unfortunately water leaks out of the barrel at the same time. The table below shows the rate of inflow, $I(t)$, and the rate of outflow, $O(t)$, at five minute intervals over first thirty minutes of the downpour. At the beginning of the downpour the barrel held 25 litres of water.

t, min	0	5	10	15	20	25	30
$I(t)$, L/min	2.0	2.7	3.6	3.0	2.4	1.5	0.5
$O(t)$, L/min	0.5	0.8	1.0	1.5	1.9	2.2	2.5

[2] (a) Write an expression involving a definite integral in terms of the functions $I(t)$ and $O(t)$ that gives the total amount of water in the barrel at the end of the thirty minute period.

Solution

Using the Net Change Theorem together with the fact that the net rate of change of the volume of water in the rain barrel is $I(t) - O(t)$, the net change of the volume $V(t)$ of water in the barrel over the first 30 minutes is

$$V(30) - V(0) = \int_0^{30} [I(t) - O(t)]\, \mathrm{d}t \Rightarrow V(30) = 25 + \int_0^{30} [I(t) - O(t)]\, \mathrm{d}t$$

[3] (b) Use the Trapezoid Rule with $n = 3$ to estimate the value of the definite integral in part (a) above and give an estimate of the total amount of water in the barrel at the end of the thirty minute period.

Solution

With $n = 3$ the value of Δt is $\Delta t = 10$. The $n = 3$ Trapezoid Rule estimate of the integral above is

$$\int_0^{30} R(t)\, \mathrm{d}t \approx T_3 = \tfrac{1}{2}(10)[R(0) + 2R(10) + 2R(20) + R(30)]$$
$$= 5(1.5 + 2(2.6) + 2(0.5) - 2.0) = 5(5.7) = 28.5$$

So the $n = 3$ Trapezoid Rule estimate of the total amount of water in the tank after the first 30 minutes is

$$V(30) = 25 + 28.5 = 53.5\,\mathrm{L}$$

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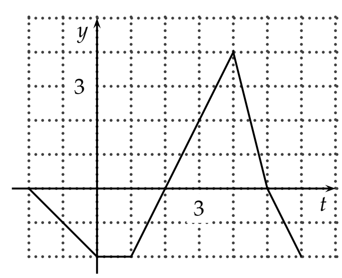
8. Let $Q(x) = \int_{-2}^{x} h(t)\, \mathrm{d}t$ where $-2 \le x \le 6$ and the graph of h is as shown.



[2] (a) Find $Q(-2)$ and $Q(6)$.

Solution

The values are

$$Q(-2) = \int_{-2}^{-2} h(t)\, \mathrm{d}t = 0$$
$$Q(6) = \int_{-2}^{6} h(t)\, \mathrm{d}t = -2 - 2 - 1 + 4 + 2 - 1 = 0$$

where the value of the second integral is obtained by finding the areas between the graph of h and the t-axis. The area is made of triangles and rectangles, with the areas given above.

Problem 8 continued ...

...Problem 8 continued

[3] (b) Determine the intervals over which Q is increasing and decreasing and classify any critical points of Q as a local maximum, local minimum, or neither.

Solution

The function Q is an antiderivative of h, so that $Q'(x) = h(x)$. Hence, Q is increasing where $h(x) > 0$ and decreasing where $h(x) < 0$. Therefore,

$$\begin{aligned} Q \text{ increasing in the interval } & (2,5) \\ Q \text{ decreasing in the interval } & (-2,2) \cup (5,6) \end{aligned}$$

There are critical numbers at $x = 2$ and $x = 5$. Recall that endpoints cannot be critical numbers. By the First Derivative Test the critical number at $x = 2$ is local minimum, since Q changes from decreasing to increasing at $x = 2$, and the critical number at $x = 5$ is local maximum, since Q changes from increasing to decreasing at $x = 5$.

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9. Evaluate each integral. Give the **exact** value of any definite integral.

[3] (a) $\displaystyle\int t^2 \left(3 + 4t^3\right)^8 \,\mathrm{d}t.$

Solution

Make the substitution

$$u = 3 + 4t^3 \Rightarrow \mathrm{d}u = 12t^2 \mathrm{d}t$$

Then the integral becomes

$$\begin{aligned} \int t^2 \left(3 + 4t^3\right)^8 \mathrm{d}t &= \tfrac{1}{12} \int \left(3 + 4t^3\right)^8 (12\mathrm{d}t) = \tfrac{1}{12} \int u^8 \,\mathrm{d}u \\ &= \tfrac{1}{12} \left(\tfrac{1}{9} u^9\right) + C = \tfrac{1}{108} \left(3 + 4t^3\right)^9 + C \end{aligned}$$

[3] (b) $\displaystyle\int_0^{\pi/4} \sec^2\theta \sqrt{1 + \tan\theta} \,\mathrm{d}\theta.$

Solution

Make the substitution

$$u = 1 + \tan\theta \Rightarrow \mathrm{d}u = \sec^2\theta \mathrm{d}\theta$$

Changing the limits of integration

$$\begin{aligned} \theta = \tfrac{1}{4}\pi &\Rightarrow u = 2 \\ \theta = 0 &\Rightarrow u = 1 \end{aligned}$$

Then the integral becomes

$$\begin{aligned} \int_0^{\pi/4} \sec^2\theta \sqrt{1 + \tan\theta} \,\mathrm{d}\theta &= \int_0^{\pi/4} \sqrt{1 + \tan\theta} \left(\sec^2\theta \,\mathrm{d}\theta\right) \\ &= \int_1^2 \sqrt{u} \,\mathrm{d}u = \tfrac{2}{3} \left(u^{3/2}\right)\Big|_1^2 = \tfrac{2}{3} \left(2\sqrt{2} - 1\right) \end{aligned}$$

Problem 9 continued ...

...Problem 9 continued

[3] (c) $\int_1^6 \frac{x+1}{\sqrt{x+3}}\,\mathrm{d}x$ using the substitution $z = \sqrt{x+3}$.

Solution

Making the given substitution

$$z = \sqrt{x+3} \Rightarrow x = z^2 - 3 \Rightarrow \mathrm{d}x = 2z\mathrm{d}z$$

Changing the limits of integration

$$x = 6 \Rightarrow z = 3$$
$$x = 1 \Rightarrow z = 2$$

Then the integral is

$$\begin{aligned}\int_1^6 \frac{x+1}{\sqrt{x+3}}\,\mathrm{d}x &= \int_2^3 \frac{z^2-2}{z}\,(2z\mathrm{d}z)\\ &= 2\int_2^3 \left(z^2-2\right)\,\mathrm{d}z = 2\left(\tfrac{1}{3}z^3 - 2z\right)\Big|_2^3\\ &= 2\left(\tfrac{1}{3}(27) - 2(3)\right) - 2\left(\tfrac{1}{3}(8) - 2(2)\right) = \tfrac{26}{3}\end{aligned}$$

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[4] 10. Use whichever of the sum formulas

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{and} \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

that are necessary to write the Riemann sum in Problem 3 without using summation notation, and evaluate an appropriate limit to find the value of the definite integral $\int_0^2 x^3\,\mathrm{d}x$.

Solution

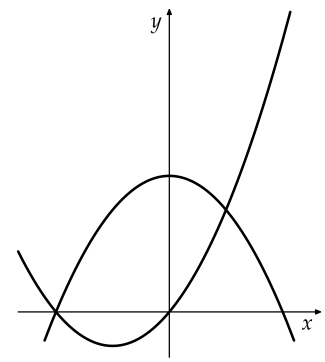
From Problem 3

$$\int_0^2 x^3\,\mathrm{d}x = \lim_{n\to\infty}\left(\frac{16}{n^4}\right)\sum_{i=1}^n i^3$$

Using the third formula above gives

$$\begin{aligned}\int_0^2 x^3\,\mathrm{d}x &= \lim_{n\to\infty}\left(\frac{16}{n^4}\right)\left[\frac{n^2(n+1)^2}{4}\right]\\ &= 4\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = 4\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 4\end{aligned}$$

[2] 11. (a) The graphs of the functions $y = x^2 + 3x$ and $y = 9 - x^2$ are shown. Label each of the graphs indicating which function is which and shade the portion of the diagram showing the region bounded by the graphs of these two functions over the interval $-3 \leq x \leq 3$.



Solution

The region is shaded in the graph shown.

[3] (b) Set up one or more definite integrals giving the area of the region bounded by the graphs of the functions $y = x^2 + 3x$ and $y = 9 - x^2$ over the interval $-3 \leq x \leq 3$. In setting up the integral(s) draw at least one vertical typical approximating rectangle on the graph above and give an expression for each approximating rectangle.

Solution

The points of intersection of the two graphs are given by

$$x^2 + 3x = 9 - x^2 \Rightarrow 2x^2 + 3x - 9 = (2x - 3)(x + 3) = 0$$

So the intersection points are at $x = -3$ and $x = \frac{3}{2}$. So there are two separate region between the curves for $-3 \leq x \leq 3$. The areas of the vertical typical approximating rectangles for the two regions are:

$$-3 \leq x < \tfrac{3}{2}: \quad \Delta A = \left(9 - x^2 - x^2 - 3x\right)\Delta x = \left(9 - 3x - 2x^2\right)\Delta x$$
$$\tfrac{3}{2} < x \leq 3: \quad \Delta A = \left(x^2 + 3x - 9 + x^2\right)\Delta x = \left(2x^2 + 3x - 9\right)\Delta x$$

So the area is given by

$$A = \int_{-3}^{3/2} \left(9 - 3x - 2x^2\right) \mathrm{d}x + \int_{3/2}^{3} \left(2x^2 + 3x - 9\right) \mathrm{d}x$$